Serendipity Virtual Element Methods

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Polytopal Element Methods in Mathematics and Engineering

Atlanta, October 26-th, 2015

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   - The operator $\mathcal{R}_S$
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6. Numerical results
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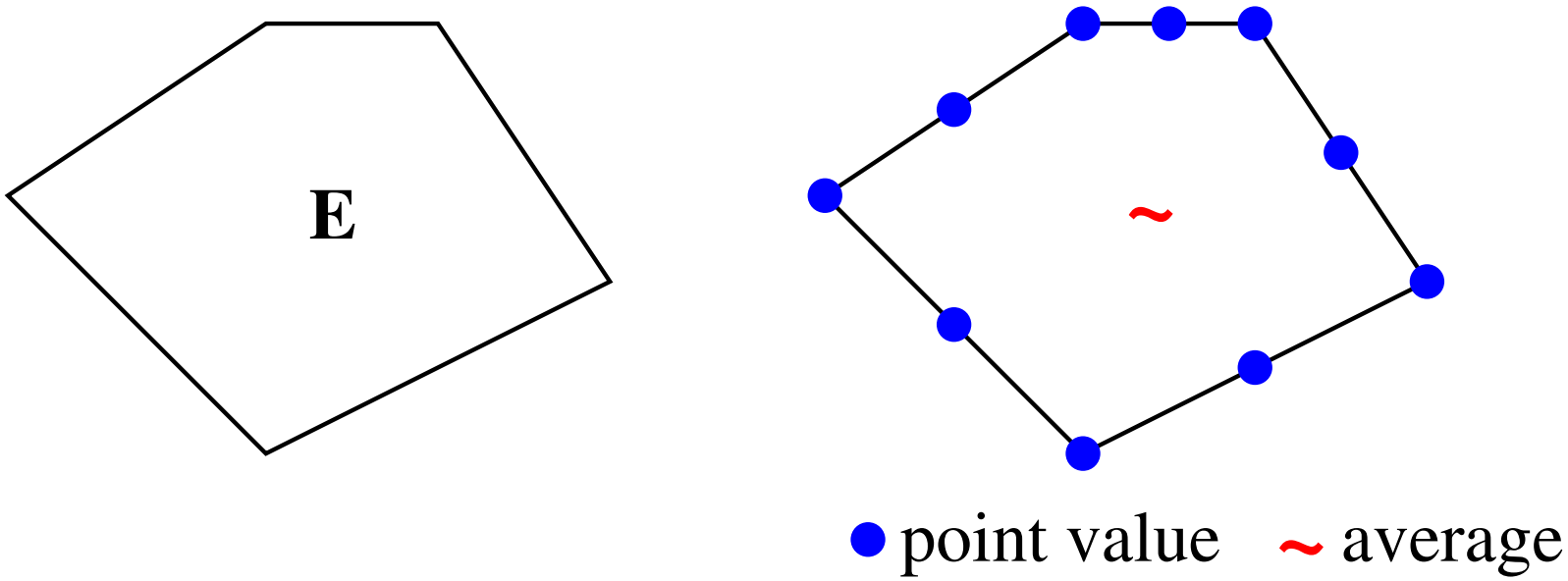
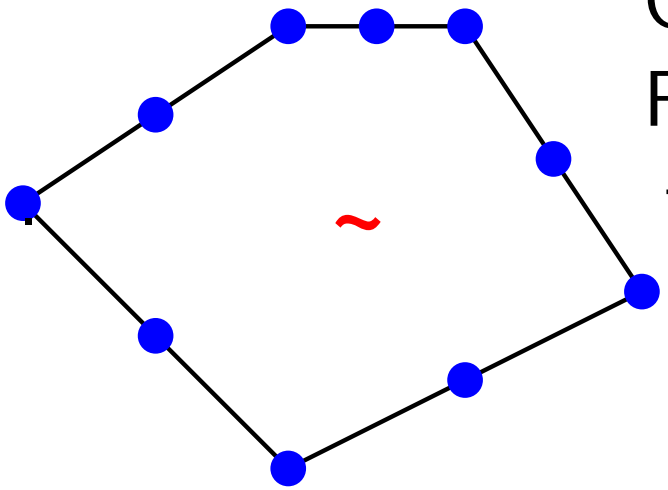
In the beginning...MFD

Assume that: E = pentagon and we want $\mathbb{P}_2$-accuracy.



We consider 11 degrees of freedom (values at vertexes and midpoints, plus the average on E). For every $v \in C^0(\bar{E})$ we define $\mathbf{D}v \in \mathbb{R}^{11}$ to be the nodal values and the average of v: $\mathbf{D}v = \Big((Dv)_1, \ldots, (Dv)_{10}, (Dv)_{11}\Big)$

MFD-2



Our (local) discrete space will be $\mathbb{R}^{11}$
For $\mathbf{P}$ and $\mathbf{Q}$ in $\mathbb{R}^{11}$ we want $[\mathbf{P}, \mathbf{Q}]_E$
to mimic, say

$$\int_E \nabla p \cdot \nabla q \, \mathrm{d}E$$

If p is (**any**) function p.w. $\mathbb{P}_2$ on ∂E, with $\mathbf{D}p = \mathbf{P}$ and q_2 a **polynomial** of degree ≤ 2 such that $\mathbf{D}q_2 = \mathbf{Q}$ then

$$\int_E \nabla p \cdot \nabla q_2 = -\int_E p \, \Delta q_2 + \int_{\partial E} p \frac{\partial q_2}{\partial n} \simeq -\mathbf{P}_{11} \int_E \Delta q_2 + \mathit{Simpson}$$

can be computed without knowing p (but only $\mathbf{P} = \mathbf{D}p$).

MFD-3

According to the previous slide, we can compute

$$[\mathbf{D}p, \mathbf{D}q]_E \simeq \int_E \nabla p \cdot \nabla q \mathrm{d}E$$

using **only** the values of $\mathbf{D}p$ and $\mathbf{D}q$ in $\mathbb{R}^{11}$ whenever either p or q is a polynomial of degree ≤ 2.
This obviously defines an 6×11 matrix R, from $\mathbb{R}^{11}$ to $\mathbb{R}^6$ that gives the values of $[\mathbf{P}, \mathbf{D}(q_2)]_E$ for $\mathbf{P} \in \mathbb{R}^{11}$ and $q_2 \in \mathbb{P}_2 \sim \mathbb{R}^6$. Then we can define a sort of *scalar product* in $\mathbb{R}^{11}$, for all $\mathbf{P}$ and all $\mathbf{Q}$ in $\mathbb{R}^{11}$, as

$$[\mathbf{P}, \mathbf{Q}]_E := (R\mathbf{P})^T \cdot (R\mathbf{Q}) + \text{ Stabilization (on the kernel of } R)$$

On the other side...FdV-based methods

"Darcy problem": $\mathbf{u} = -\nabla p$, $div\mathbf{u} = f$ + B.C.
From now on: $p = 0$ on the boundary.
Following B. M. Fraeijs de Veubeke (**1965**) we observe that: given an approximation λ_h of p at the inter-element boundaries, and *another* approximation p_h of p inside each element we can deduce an approximation $\mathbf{u}_h$ of $\mathbf{u}$ by requiring, in each element E

$$\int_E \mathbf{u}_h \cdot \mathbf{v} = \int_E p_h \, \mathrm{div}\mathbf{v} + \int_{\partial E} \lambda_h \, (\mathbf{v} \cdot \mathbf{n}) \text{ for all } \mathbf{v}.$$

This defines an *approximate gradient*

$$\mathbf{G_h} : \ (\lambda_h, p_h) \to \mathbf{u}_h = \mathbf{G_h}(\lambda_h, p_h)$$

Use of FdV

Darcy problem: $\mathbf{u} = -\nabla p$, $div\mathbf{u} = f$, $p = 0$ at $\partial\Omega$

$$\int_E \mathbf{u}_h \cdot \mathbf{v} = \int_E p_h \, \mathrm{div}\mathbf{v} + \int_{\partial E} \lambda_h \, (\mathbf{v} \cdot \mathbf{n}) \text{ for all } \mathbf{v}.$$

You can then add a discretized *conservation equation*

$$\int_E \mathrm{div}\mathbf{u}_h \, q = \int_E f \, q \text{ for all } q.$$

Then you must **close the system**, with *as many equations as there are λ_h's*, requiring "continuity" (for $\mathbf{u}_h \cdot \mathbf{n}$ or for p_h, or for a combination of the two) at the interelement boundaries. You can get **zillions** of methods.

Approximate Gradient methods

Remember that given a λ_h at the inter-element edges (or faces), and a p_h inside each element, we can define an *approximate gradient*

$$\mathbf{G_h} : \ (\lambda_h, p_h) \to \mathbf{u}_h = \mathbf{G_h}(\lambda_h, p_h)$$

by requiring, in each element E

$$\int_E \mathbf{G_h} \cdot \mathbf{v} = \int_E p_h \, \mathrm{div}\mathbf{v} + \int_{\partial E} \lambda_h \, (\mathbf{v} \cdot \mathbf{n}) \text{ for all } \mathbf{v}$$

where $\mathbf{v}$ ranges over the space where you look for $\mathbf{G_h}$. Then you look for a pair (λ_h, p_h) such that

$$\sum_E \int_E \mathbf{G_h}(\lambda_h, p_h) \cdot \mathbf{G_h}(\mu_h, q_h) = \int_\Omega f \, q_h \text{ for all } (\lambda_h, q_h).$$

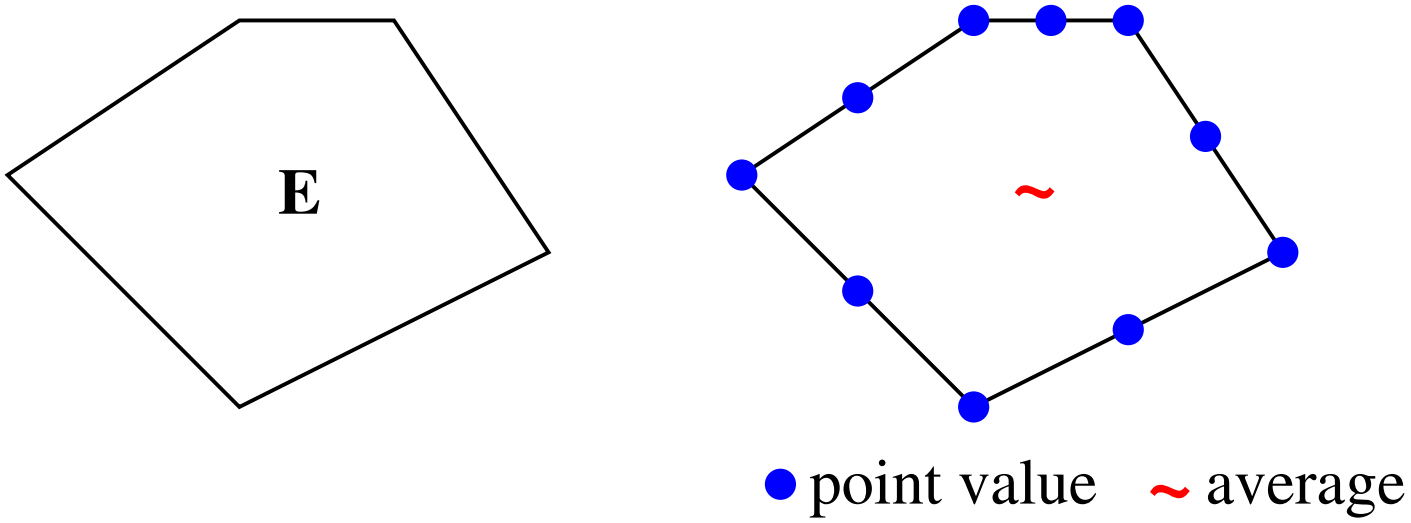
Approximate Gradient methods-2

Actually, you also need to add a stabilizing term *like*

$$\sum_E \int_E \mathbf{G_h}(\lambda_h, p_h) \cdot \mathbf{G_h}(\mu_h, q_h)$$
$$+ C\, h^{-1} \sum_e \int_e (\lambda_h - p_h)(\mu_h - q_h) = \int_\Omega f\, q_h, \quad \forall \mu_h\, \forall q_h.$$

Various methods distinguish themselves for *the space where you look for the approximate gradient* and for *the type of stabilization used*. You can get **zillions** of methods

Basic idea of VEMs



We take again 11 degrees of freedom (values at vertexes and midpoints, + the average on E). We define the space

$$V_E := \{v \,|\, \text{such that } v_{|e} \in \mathbb{P}_2(e)\ \forall \text{ edge } e, \text{ and } \Delta v \in \mathbb{P}_0(E)\}$$

It is easy to see that our 11 d.o.f.s are V_E-*unisolvent*.

Basic idea of VEMs-2

To every $v \in V_E$ we associate $\Pi_2^E v \in \mathbb{P}_2(E)$ defined by

$$\int_E \nabla(\Pi_2^E v - v) \cdot \nabla q_2 = 0 \text{ for all } q_2 \in \mathbb{P}_2(E)$$

$$\int_E (\Pi_2^E v - v) \mathrm{d}E = 0.$$

Note that the quantity (= right-hand side)

$$\int_E \nabla v \cdot \nabla q_2 = -\Delta q_2 \int_E v + \int_{\partial E} v \frac{\partial q_2}{\partial n}$$

is computable (from the dofs) $\forall v \in V_E$ and $\forall q_2 \in \mathbb{P}_2$.

Basic idea of VEMs-3

Discretized problem: *find* $p_h \in V$ *such that*

$$\sum_E \int_E \nabla \Pi_2^E p_h \cdot \nabla \Pi_2^E q_h + S(p_h, q_h) = \sum_E \int_E f \, \Pi_2^E q_h \, \forall q_h \in V$$

where the *stabilizing term* $S(p_h, q_h)$ can be taken as

$$S(p_h, q_h) := \sum_E \left(\mathbf{D}(q_h - \Pi_2^E q_h) \right)^T \cdot \left(\mathbf{D}(p_h - \Pi_2^E p_h) \right)$$

where $\mathbf{D}$ is the *degrees of freedom vector* defined before.

Panorama

In summary:

- AD-HOC functions (e.g. Rational, Baricentric,...); one field and numerical integration
- DG: One field, discontinuous
- MFD: Only dofs, no functions
- HDG WG HHO: Two/three polynomial fields $(\lambda, p, (\mathbf{u}))$
- VEM: One field, solution of a PDE.

Original VEM-General order

For k integer ≥ 1 we define

$$V_k(E) = \{\varphi \in C^0(\bar{E}) : \varphi_{|e} \in \mathbb{P}_k(e) \forall \text{ edge } e,\ \Delta\varphi \in \mathbb{P}_{k-2}(E)\}.$$

The degrees of freedom in $V_k(E)$ are taken as

- the values of φ at the vertices,
- $\displaystyle\int_e \varphi\, q \mathrm{d}s \quad$ for all $q \in \mathbb{P}_{k-2}(e) \quad \forall$ edge e,
- $\displaystyle\int_E \varphi\, q \mathrm{d}E \quad$ for all $q \in \mathbb{P}_{k-2}(E)$.

It is immediate to verify that the degrees of freedom are unisolvent

Classical VEM-General case

For k integer ≥ 1 and k_Δ integer ≥ -1 we define

$$V_{k,k_\Delta}(E) = \{\varphi \in C^0(\bar{E}) : \varphi_{|e} \in \mathbb{P}_k(e) \forall \text{ edge } e,\ \Delta\varphi \in \mathbb{P}_{k_\Delta}(E)\}$$

The degrees of freedom in $V_k(E)$ are taken as

- the values of φ at the vertices,
- $\displaystyle\int_e \varphi\, q \mathrm{d}s \quad$ for all $q \in \mathbb{P}_{k-2}(e) \quad \forall$ edge e,
- $\displaystyle\int_E \varphi\, q \mathrm{d}E \quad$ for all $q \in \mathbb{P}_{k_\Delta}(E)$.

It is immediate to verify that the degrees of freedom are unisolvent. In general it is better to take $k_\Delta \geq k - 2$.

Classical VEM-General case -3D

For k integer ≥ 1,and k_Δ, k_f integers ≥ -1 we define

$$V_{k,k_f,k_\Delta}(E) = \{\varphi \in C^0(\bar{E}) : \varphi_{|f} \in V_{k,k_f} \forall \text{ face } f,\ \Delta\varphi \in \mathbb{P}_{k_\Delta}(E)\}$$

The following degrees of freedom are unisolvent

- the values of φ at the vertices,
- $\displaystyle\int_e \varphi\, q \mathrm{d}s \quad$ for all $q \in \mathbb{P}_{k-2}(e) \quad \forall$ edge e,
- $\displaystyle\int_f \varphi\, q \mathrm{d}s \quad$ for all $q \in \mathbb{P}_{k_f}(f) \quad \forall$ face f,
- $\displaystyle\int_E \varphi\, q \mathrm{d}E \quad$ for all $q \in \mathbb{P}_{k_\Delta}(E)$.

VEM and FEM on triangles: degrees of freedom

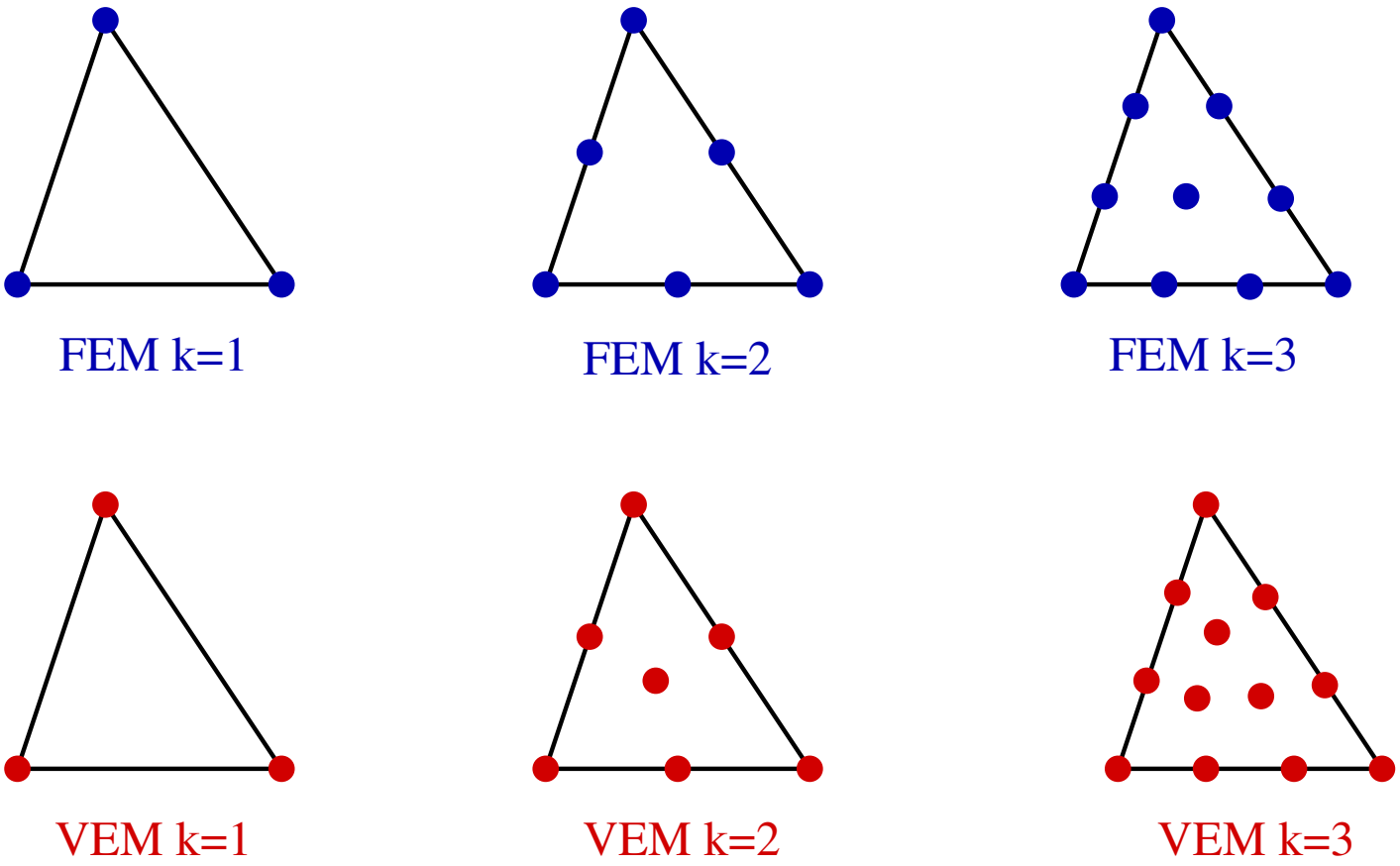


Figure: Triangles: Classical FEM and Original VEM

VEM and FEM on quads: degrees of freedom

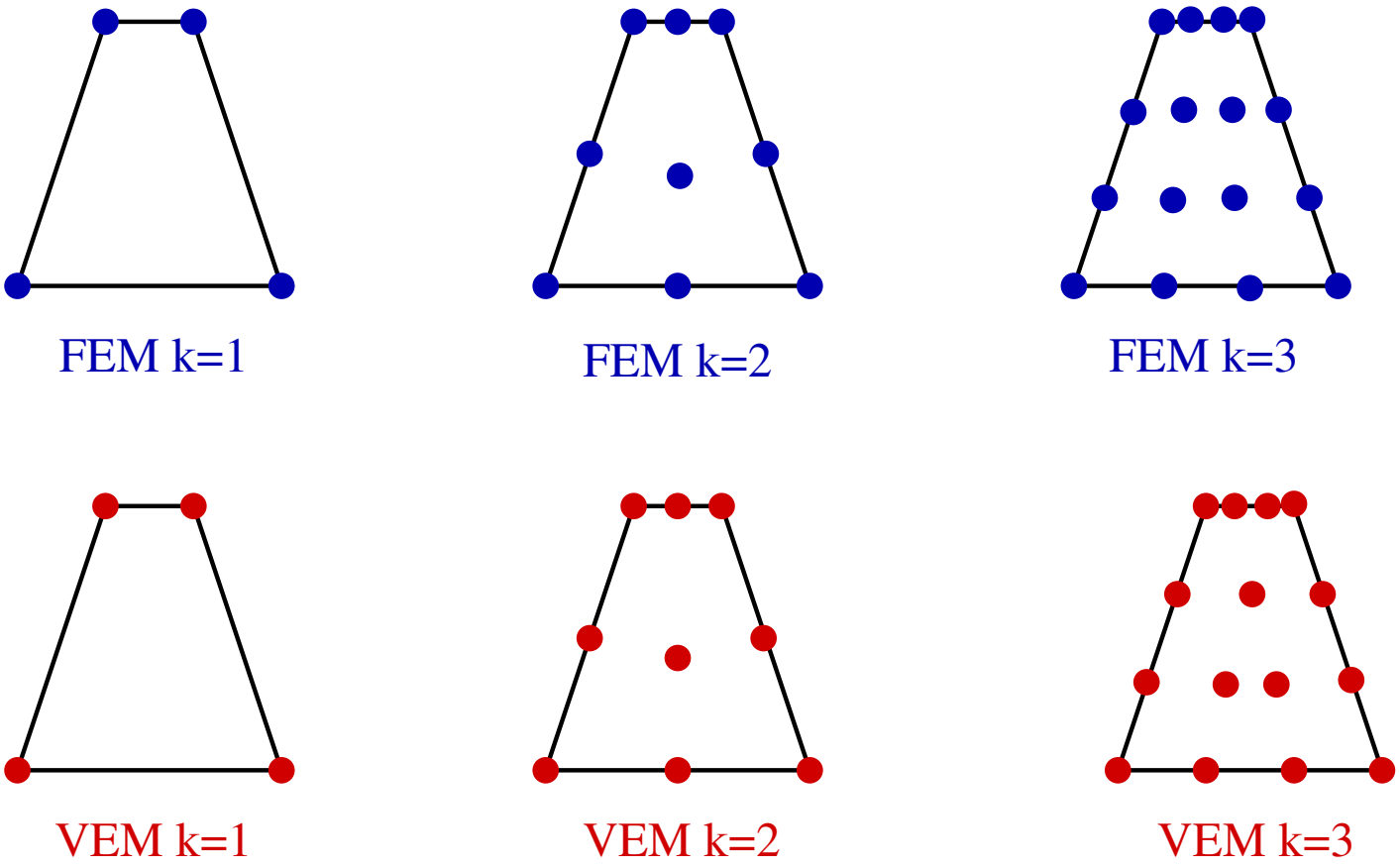


Figure: Quads: Classical FEM and Original VEM

The **reduced** dofs

We choose an S with $(k+1)(k+2)/2 \leq S \leq N_E$ (N_E=dimensiion ov $V_{k,k}$), and *assume* that the degrees of freedom of $V_{k,k}$ are ordered in such a way that the first S

$$\delta_1,\ \delta_2,\ \ldots \delta_S$$

have the following properties:

- $(\mathscr{B})$ They include all the *boundary dofs*
- $(\mathscr{S})$ $\forall p_k \in \mathbb{P}_k(E)$ we have:

$$\{\delta_1(p_k) = \delta_2(p_k) = \ldots = \delta_S(p_k) = 0\} \Rightarrow \{p_k \equiv 0\}$$

Property $(\mathscr{B})$ is there to ensure *conformity*, and is easy.

The **reduced** dofs - Examples

Once $(\mathscr{B})$ is satisfied, you know that a polynomial p_k that satisfies $\{\delta_1(p_k) = \delta_2(p_k) = \ldots = \delta_S(p_k) = 0\}$ must be identically zero on the boundary. Let's deal with $(\mathscr{S})$.

To get Property $(\mathscr{S})$, on top of the bounday dofs:

- on a **triangle**, you must include as many internal dofs as the dimension of $\mathbb{P}_{k-3}$,
- on a **square**, you must include as many internal dofs as the dimension of $\mathbb{P}_{k-4}$,
- on a regular **n-gon** you must include as many internal dofs as the dimension of $\mathbb{P}_{k-n}$.

In general (even on very distorted polygons): **you must have as many internal dofs as there are $\mathbb{P}_k$-bubbles**.

VEMS on quads

As we are allowing dramatic distortions, the number of $\mathbb{P}_k$-bubbles depends also on the geometry of the element. Here are some examples on quadrilaterals for $k = 4$.

Number of P_4 –bubbles on quadrilaterals

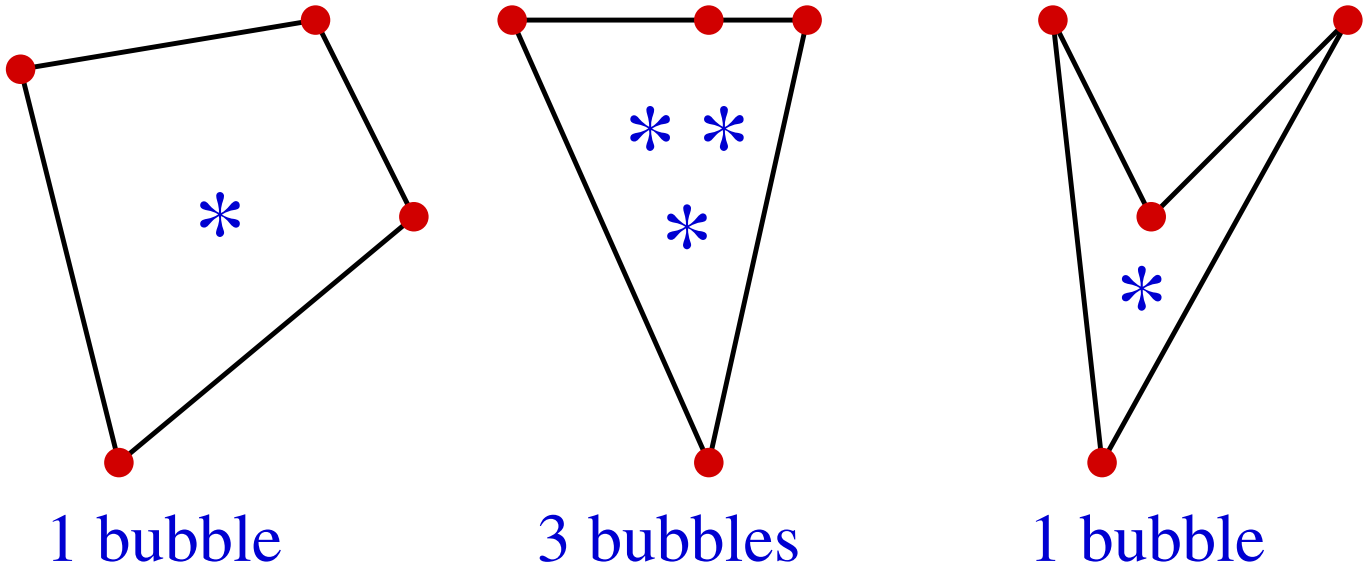


Figure: Allowed distortions for quadrilaterals

The **lazy** choice and the **stingy** choice

As we have seen, the minimum number of internal degrees of freedom that have to be kept depends on k, and on the geometry of the element. Typically: is the dimension of $\mathbb{P}_{k-\eta}$ where η is the minimum number of straight lines necessary to cover all the boundary of the element.

In practice, in the **code**, you can either check every element to compute its η (**stingy choice**), or treat every element as if it was a triangle (**lazy choice**).

One or the other choice could be preferable, depending on the circumstances.

The operator $\mathcal{D}_S$

We assume now that, for a given k, we are given a set $\delta_1, \delta_2, \ldots \delta_S$ of degrees of freedom having the properties $\mathscr{B}$ and $\mathscr{S}$, and we define **the operator $\mathcal{D}_S$**

$\mathcal{D}_S : V_{k,k}(E) \to \mathbb{R}^S$ defined by $\mathcal{D}_S \varphi := (\delta_1(\varphi), \ldots, \delta_S(\varphi))$.

Needless to say, the operator $\mathcal{D}_S$ has the properties

- $\mathcal{D}_S$ can be computed using only the d.o.f $\delta_1, \ldots, \delta_S$,
- $\mathcal{D}_S\, q = 0 \Rightarrow q = 0 \quad$ for all $q \in \mathbb{P}_k$.

$\mathscr{S} \equiv \{\mathcal{D}_S$ **is injective when applied to** $\mathbb{P}_k\}$

The operator $\mathcal{R}$

We are now going to use $\mathcal{D}_S$ to construct another operator, $\mathcal{R}_S$, as follows: $\forall\, \varphi \in V_{k,k}(E)$ we define $\mathcal{R}_S\, \varphi \in \mathbb{P}_k$ by

$$(*) \qquad (\mathcal{D}_S \mathcal{R}_S \varphi - \mathcal{D}_S \varphi, \mathcal{D}_S q)_{\mathbb{R}^s} = 0 \quad \forall q \in \mathbb{P}_k,$$

where $(\cdot\,,\,\cdot)_{\mathbb{R}^s}$ is the "*Euclidean scalar product*" in $\mathbb{R}^S$.
Property $\mathscr{S}$ ensures that the matrix

$$(\mathcal{D}_S p, \mathcal{D}_S q)_{\mathbb{R}^s} \qquad p,\ q \in \mathbb{P}_k$$

is nonsingular, so that for every φ (and hence for every r.h.s. $(\mathcal{D}_S(\varphi), \mathcal{D}_S q)_{\mathbb{R}^s}$) $(*)$ has a unique solution $\mathcal{R}_S \varphi$.

$$\mathcal{R}_S \text{ is a projector } V_{k,k} \to \mathbb{P}_k$$

The Serendipity VEM spaces

The operator $\mathcal{R}_s$ that we constructed has the properties

- $\mathcal{R}_S$ is computable using only the d.o.f. $\delta_1, ..., \delta_S$,
- $\mathcal{R}_S q_k = q_k$ for all $q_k \in \mathbb{P}_k$,

that allow us to construct the Serendipity VEM space

$$V_k^S(E) = \{\varphi \in V_{k,k}(E) \text{ s.t.} \delta_r(\varphi) = \delta_r(\mathcal{R}_S \varphi) \, \forall r = S+1, .., N_E\}$$

From the first S dofs we can compute $\mathcal{R}_S$ and then using $\mathcal{R}_S$ we compute all the others. And $\mathbb{P}_k \subseteq V_k^S$!

FEM and S-VEM: triangles

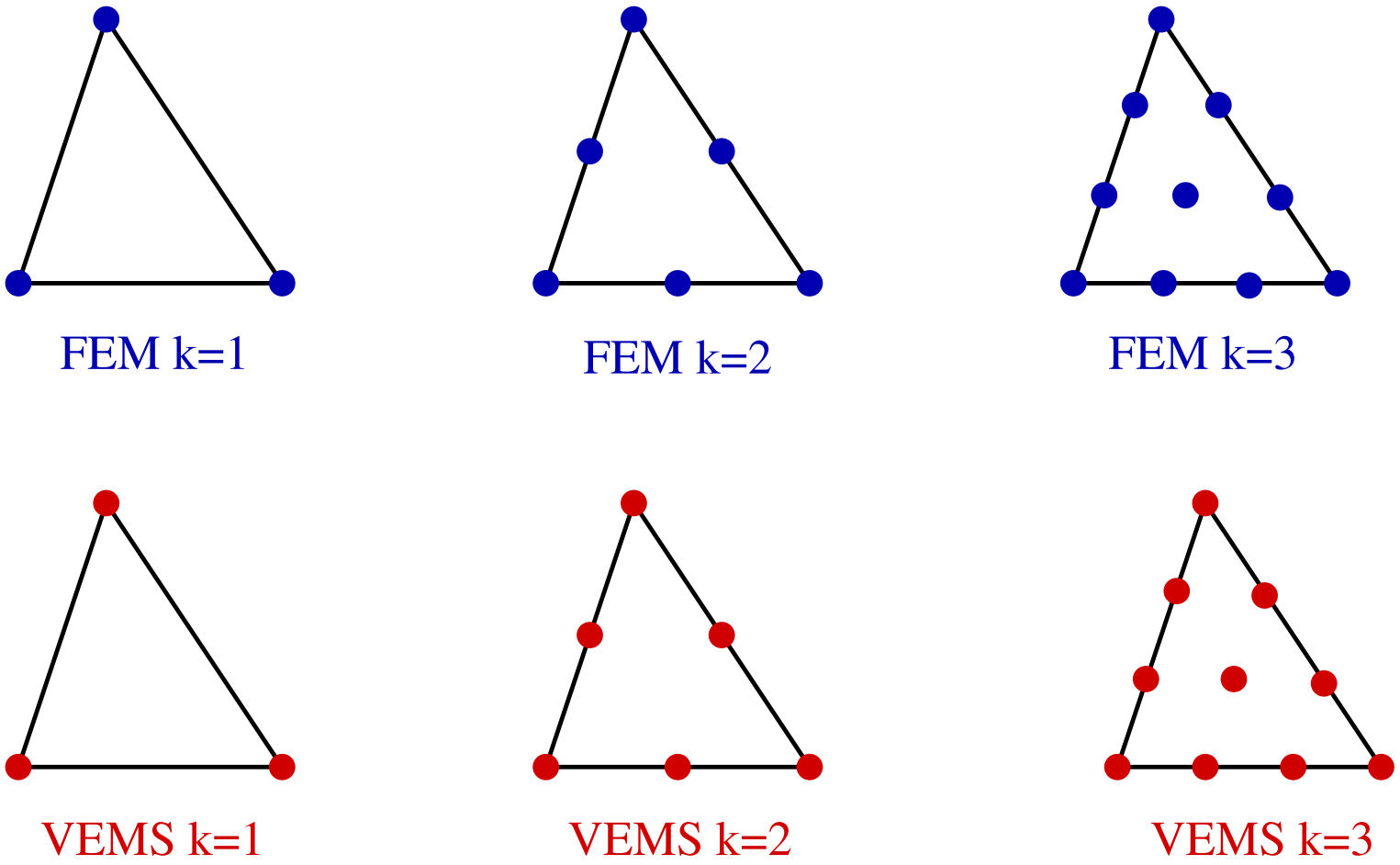


Figure: Triangles: Classical FEM and Serendipity VEM

S-FEMS and S-VEM: quads

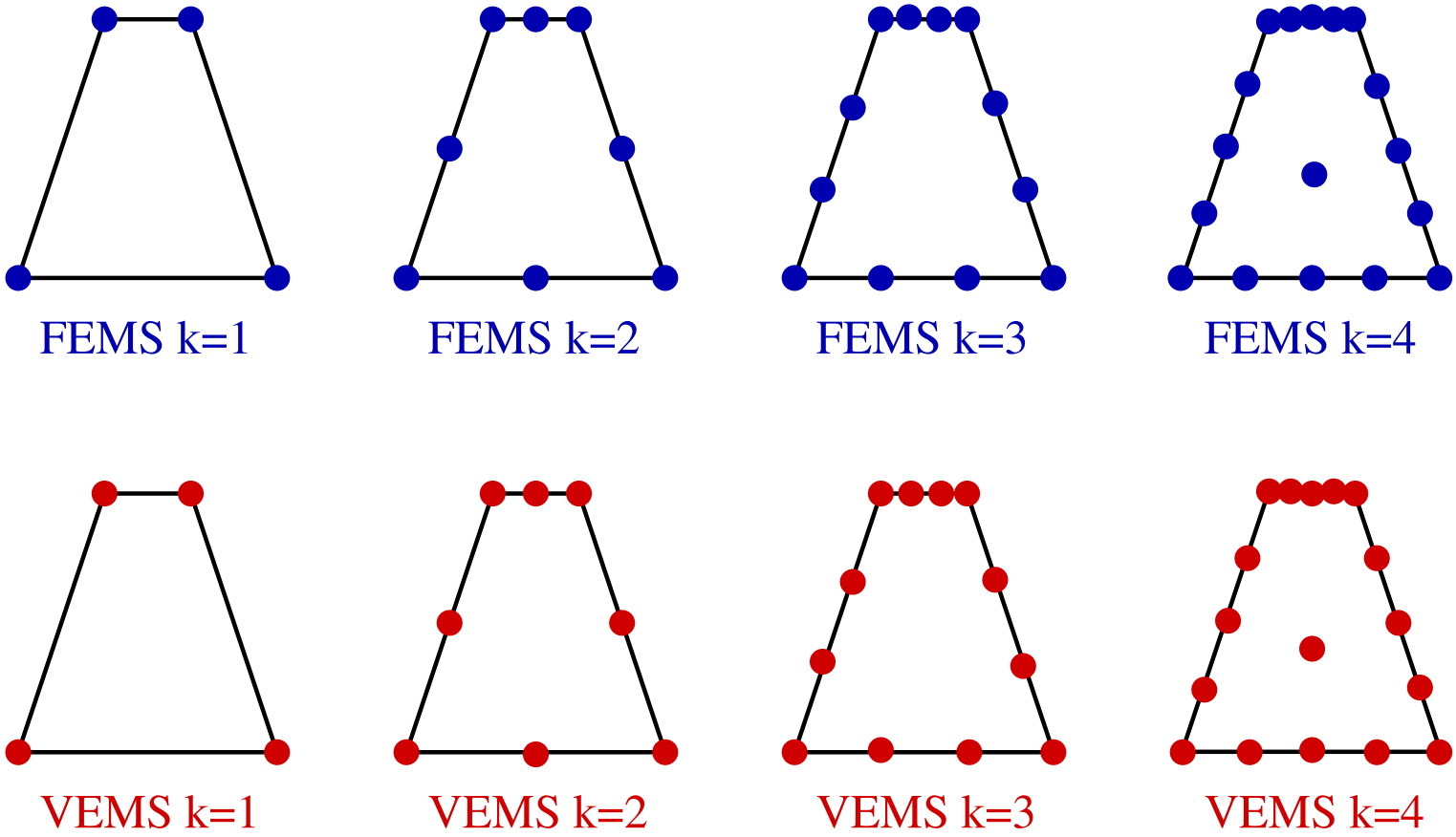


Figure: Quads: S-FEM (Arnold-Awanou) and S-VEM

Numerical results: Meshes

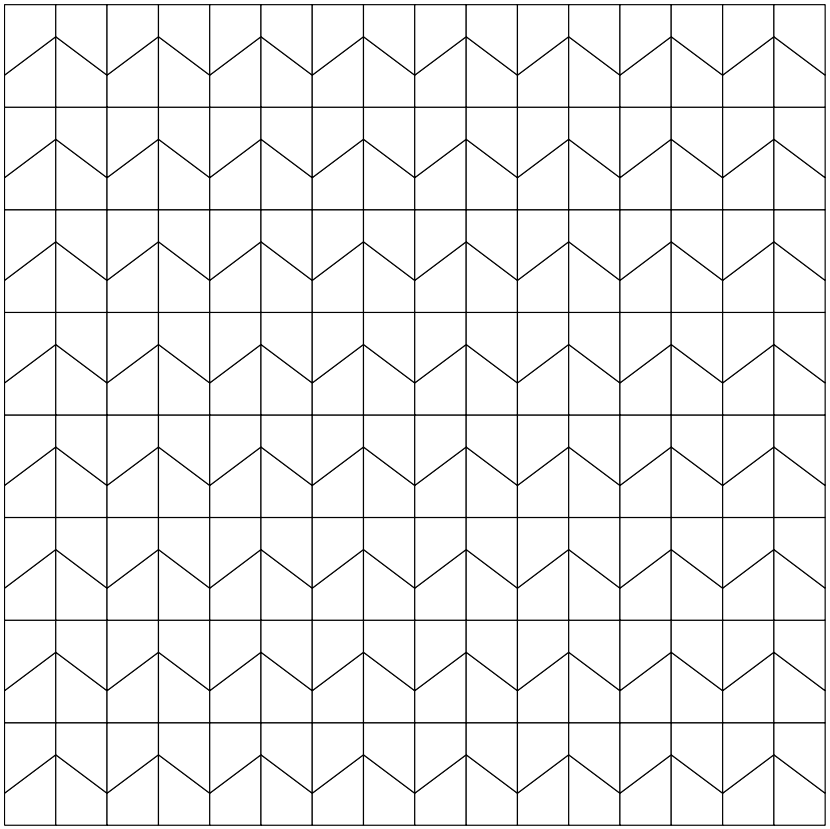


Figure: Trapezoidal mesh

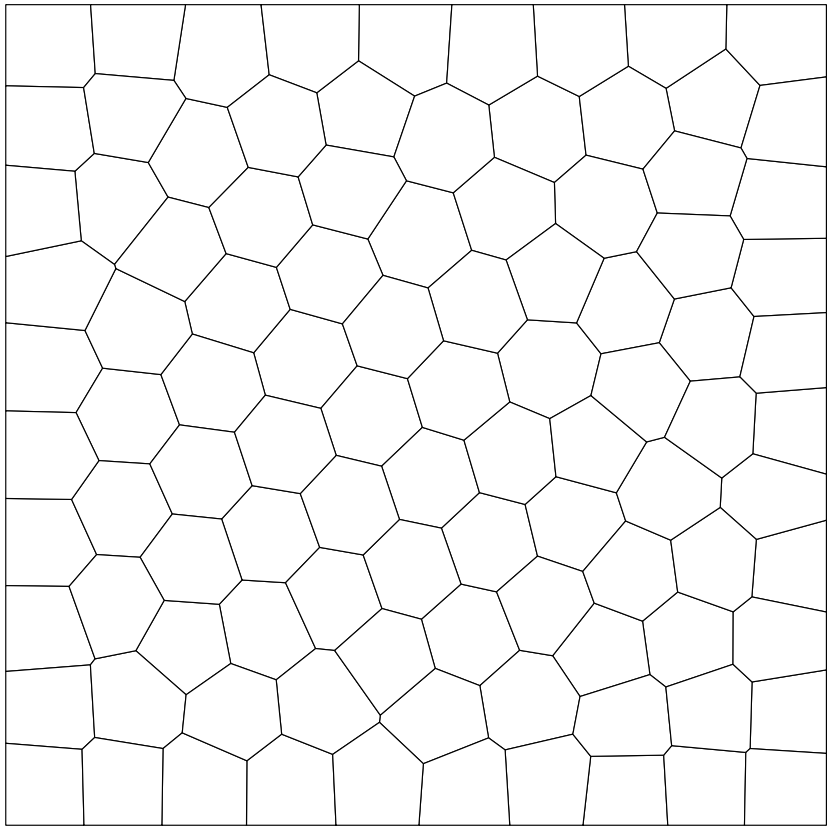


Figure: Voronoi-Lloyd mesh

Numerical results: Q_k-FEM, S-FEM, and S-VEM on quads

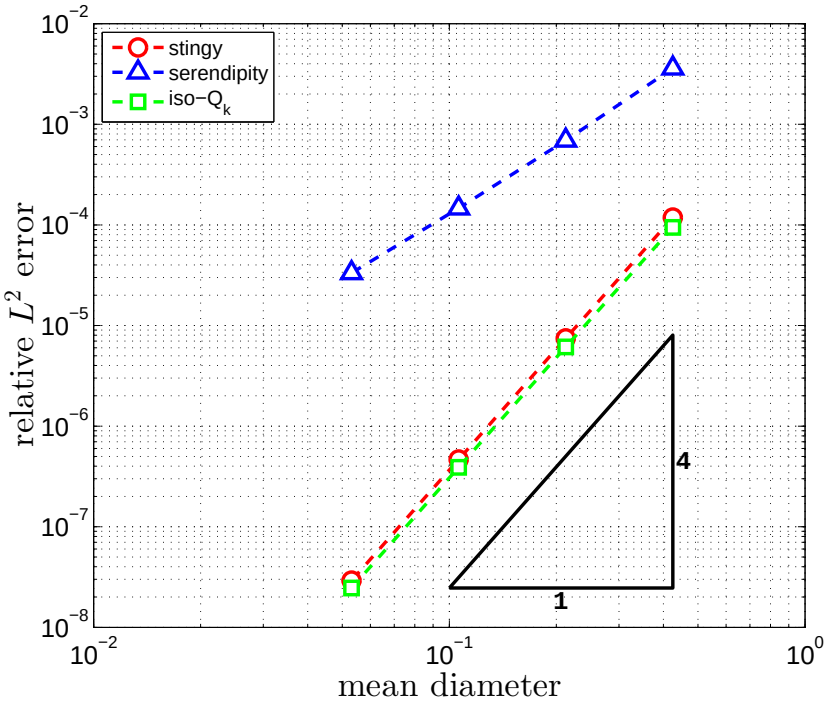


Figure: Trapezoidal mesh $k = 3$

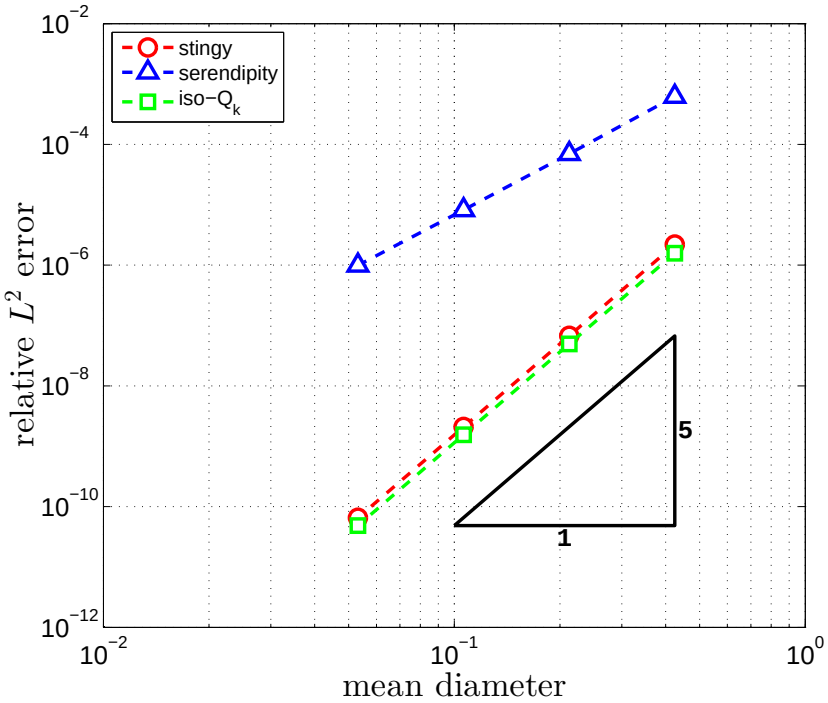


Figure: Trapezoidal mesh $k = 4$

Classical VEM and S-VEM (stingy, lazy) on V-Lloyd

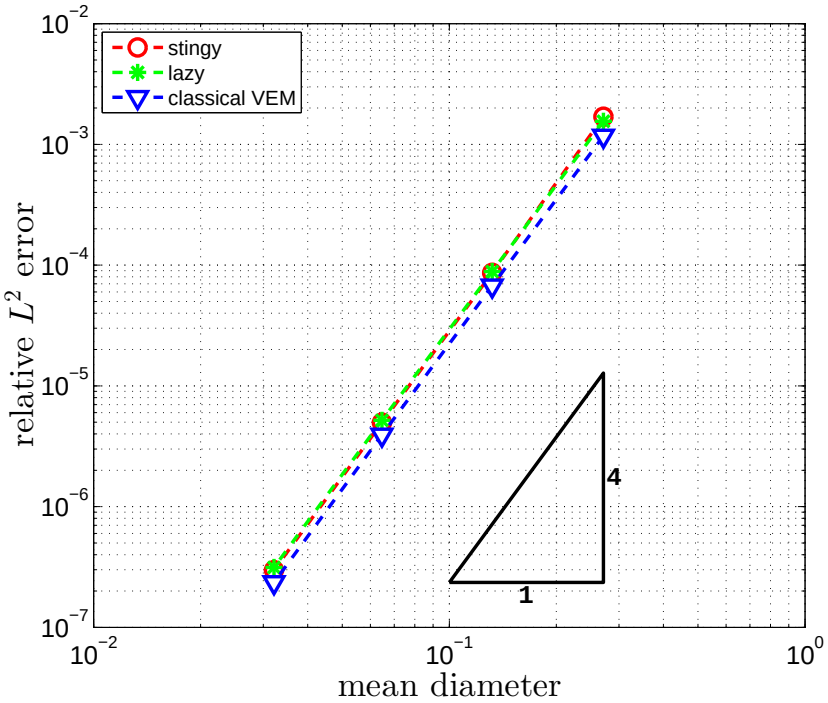


Figure: Voronoi-Lloyd mesh $k = 3$

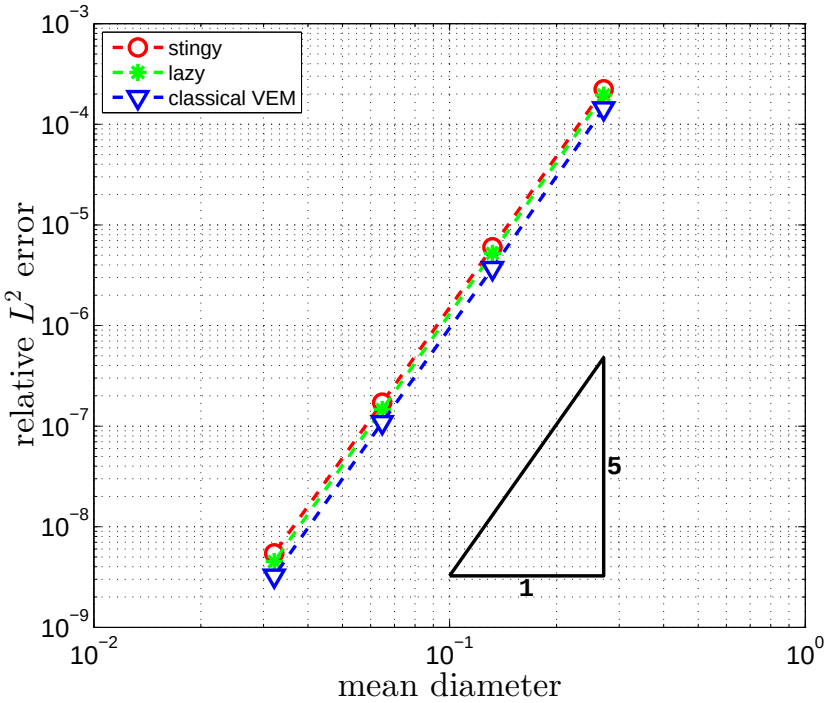


Figure: Voronoi-Lloyd $k = 4$

Final comments and conclusions

- Serendipity Virtual Elements allow a drastic reduction of the number of *internal* degrees of freedom.
- In 3 dimensions, the degrees of freedom *internal to faces* are also reduced, making a big difference.
- As the older *enhanced VEMs*, the Serendipity VEMs allow an immediate computation of the L^2-projection (of trial and test functions).
- On **triangles** we (finally!) reproduce classical FEMs.
- On **quads**, S-VEM imitate Serendipity FEM, **without paying for distortions**.
- And they preserve all the "geometric freedom" of the original VEMs.