C++ Polymorphism for Weak Galerkin (WG) Finite Element Methods on Polytopal Meshes

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Preliminary Report

2015/10/27, POEMS @ GaTech

Acknowledgements

Funding

- National Science Foundation, DMS-1419077

With help from

- Farrah Sadre-Marandi: *ColoState, now MBI (Ohio State)*
- Zhuoran Wang: *ColoState*

Outline

- Review: Basic concepts of WG finite element methods
- Some considerations of WG implementation in C++
- Case study: Lowest order WG for Darcy flow
  - on 2-dim meshes with mixed triangles, quadrilaterals thru C++ polymorphism
  - on 3-dim (general) hexahedral meshes
- Further considerations of WG implementation and applications – e.g., Dimension-independent code ?

Model Problem: Elliptic BVP

Boundary value problem (BVP) for 2nd order elliptic equation

$$\begin{cases} \nabla \cdot (-\nabla p) = f, & \mathbf{x} \in \Omega, \\ p = 0, & \mathbf{x} \in \Gamma := \partial\Omega. \end{cases} \tag{1}$$

Variational formulation: Seek $p \in H_0^1(\Omega)$ such that $\forall q \in H_0^1(\Omega)$

$$\int_\Omega \nabla p \cdot \nabla q = \int_\Omega fq.$$

Finite element methods:

- CG: $V_h \subset H_0^1(\Omega)$, No local conservation or cont. normal flux;
- DG: $V_h \not\subset H_0^1(\Omega)$, DOFs proliferation, penalty factor;
- WG: Approx. ∇p by weak gradient, Many good features!

Weak Functions and Weak Gradient

See Wang,Ye, *JCAM* (2013)

A **weak function** on an element E has 2 pieces $v = \{v^\circ, v^\partial\}$

- in interior $v^\circ \in L_2(E^\circ)$;
- on element boundary $v^\partial \in L_2(\partial E)$.

Note: v^∂ may not be the trace of v°, should a trace be defined.

For any weak function $v \in W(E)$, its **weak gradient** $\nabla_w v$ is defined (interpreted) as a linear functional on $H(\mathrm{div}, E)$:

$$\int_E (\nabla_w v)\cdot \mathbf{w} = \int_{\partial E} v^\partial (\mathbf{w}\cdot\mathbf{n}) - \int_E v^\circ (\nabla\cdot\mathbf{w}) \quad \forall \mathbf{w} \in H(\mathrm{div}, E). \quad (2)$$

Integration By Parts !

Similarly, weak curl/divergence (of vector-valued weak functions)

Weak Galerkin Essentials: Discrete Weak Gradient

Let $l, m, n \geq 0$ be integers, $V(E, n)$ a subsp. of $P^n(E)^d (d = 2, 3)$.

- A **discrete weak function** is a weak function $v = \{v^\circ, v^\partial\}$ such that $v^\circ \in P^l(E^\circ)$, $v^\partial \in P^m(\partial E)$.
- For a disc.wk.fxn. v, its **discrete weak gradient** is defined by

$$\int_E \nabla_{w,n} v \cdot \mathbf{w} = \int_{\partial E} v^\partial (\mathbf{w} \cdot \mathbf{n}) - \int_E v^\circ (\nabla \cdot \mathbf{w}) \quad \forall \mathbf{w} \in V(E, n). \tag{3}$$

So $\nabla_{w,n} v$ is a lin. comb. of basis fxns. of $V(E, n)$.

Example: (P_0, P_0, RT_0) on a triangle.

Implementation involves

- Three traditional FE spaces: $P^l(E^\circ), P^m(\partial E), V(E, n)$;
- Gram matrix of a basis for $V(E, n)$;
  Solving a small SPD lin. sys. (Cholesky factorization).

WG Element $(Q_0, P_0, RT_{[0]})$ on a Rectangle

Let $E = [x_1, x_2] \times [y_1, y_2]$ be a rectangular element.
For WG element $(Q_0, P_0, RT_{[0]})$, there are 5 WG basis functions:
– One constant function in element interior ϕ°
– One constant fxn. for each of the 4 edges $\phi_i^\partial (i = 1, 2, 3, 4)$.
Their discrete weak gradients are specified as in $RT_{[0]}$.

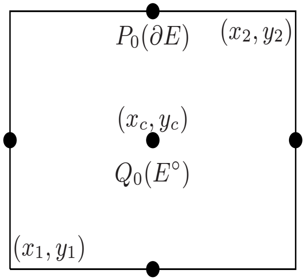


Figure: 5 basis functions for a WG $(Q_0, P_0, RT_{[0]})$ rectangular element.

WG $(Q_0, P_0, RT_{[0]})$ on Rectangle

Let $E = [x_1, x_2] \times [y_1, y_2]$ be a rectangular element.

$$\begin{cases} \nabla_{w,n}\phi^\circ = 0\mathbf{w}_1 + 0\mathbf{w}_2 + \frac{-12}{(x_2-x_1)^2}\mathbf{w}_3 + \frac{-12}{(y_2-y_1)^2}\mathbf{w}_4, \\ \nabla_{w,n}\phi_1^\partial = 0\mathbf{w}_1 + \frac{-1}{y_2-y_1}\mathbf{w}_2 + 0\mathbf{w}_3 + \frac{6}{(y_2-y_1)^2}\mathbf{w}_4, \\ \nabla_{w,n}\phi_2^\partial = \frac{1}{x_2-x_1}\mathbf{w}_1 + 0\mathbf{w}_2 + \frac{6}{(x_2-x_1)^2}\mathbf{w}_3 + 0\mathbf{w}_4, \\ \nabla_{w,n}\phi_3^\partial = 0\mathbf{w}_1 + \frac{1}{y_2-y_1}\mathbf{w}_2 + 0\mathbf{w}_3 + \frac{6}{(y_2-y_1)^2}\mathbf{w}_4, \\ \nabla_{w,n}\phi_4^\partial = \frac{-1}{x_2-x_1}\mathbf{w}_1 + 0\mathbf{w}_2 + \frac{6}{(x_2-x_1)^2}\mathbf{w}_3 + 0\mathbf{w}_4. \end{cases}$$

where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ are the normalized bas. fxn. of $RT_{[0]}(E)$:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} X \\ 0 \end{bmatrix}, \quad \mathbf{w}_4 = \begin{bmatrix} 0 \\ Y \end{bmatrix},$$

and $X = x - x_c$, $Y = y - y_c$, (x_c, y_c) element center.

WG Elements on Quadrilaterals / Hexahedra

Need for quadrilaterals or more general polygons:

- Flexibility of accommodating problem geometry
- Reducing degrees of freedom

WG elements on quadrilaterals or polygons

- WG (P_1, P_1, P_0^2), See Mu,Wang,Ye, IJNAM (2015)
- WG (P_1, P_0, P_0^2), Shown in Xiu Ye talk yesterday

Another try: WG $(Q_0, P_0, RT_{[0]})$ on quadrilaterals?

Need for Polytonal Meshes

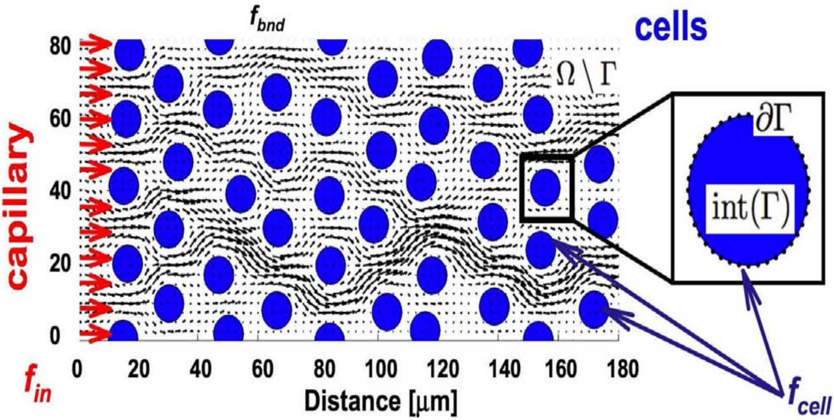


Figure: Darcy flow in the interstitial space around tumor cells. Source: Rejniak et al., *Frontiers in Oncology*, 2013.

Hexahedral Meshes instead of tetrahedral meshes

For certain complicated domains, e.g.,
– Geosci. problems; Wheeler,Xue,Yotov, *Comput.Geosci.* (2012)
– Coronary artery wall; Hossain et al., *Comput.Mech.* (2012)

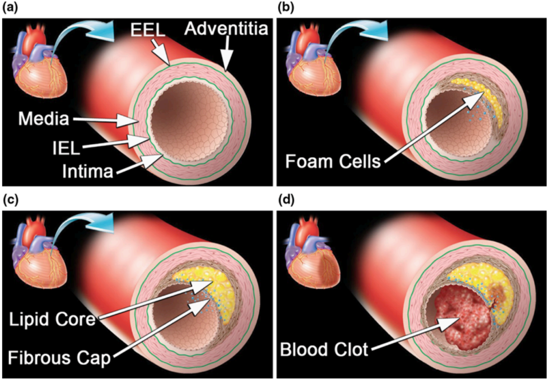


Figure: Hossain,Hossainy,Bazilevs,Calo,Hughes, *Comput.Mech.* (2012)

WGFEMs Implementation

Existing work

- Lin Mu: Matlab code
- Long Chen: *iFEM*
- Liu,Sadre-Marandi: *DarcyLite* (Matlab code)
- ...

New efforts:

C++ implementation

Ingredients of FEMs

What are involved?

1. Domain and its boundary;
2. PDEs and boundary conditions
3. Mesh and cells
4. Elements (cells being equipped with basis functions)
5. (Discrete weak) gradient / curl /div of (WG) basis functions
6. Bilinear and linear forms at the element level
   – e.g., grad-grad
7. Assembly at the mesh level;
   Mesh topology info used for global matrix sparsity pattern
   - Element vs its edges
   - Edge vs neighboring edges
   - class `SparseBlockMatrix`

Multiple Inheritance

- Inheritance from the class for $P^l(E^\circ)$,
  Using the basis functions:
  – their values,
  – Gram matrix for local L_2-projection $\mathbb{Q}^\circ$, etc.
- Inheritance from the class for $P^m(\partial E)$,
  Using the basis functions:
  – their values,
  – Gram matrix for local L_2-projection $\mathbb{Q}^\partial$, etc.
- Inheritance from the class for $V(E, n)$,
  Using the basis functions:
  – Gram matrix for solving the small SPD lin. sys. in Eqn.(3),
  – values of basis functions when flux/velocity is needed.

The classes for $P^l(E^\circ), V(E, n)$ are *derived classes* of the class for geometric cell E.

Classes for Geometric Cells

Just geometric features/properties:
– vertices, volume, outer unit normal on boundary faces, ...

Enumerations of geometric cells in 2-dim, 3-dim:

`Tri2d`	2-dim triangles
`Rect2d`	2-dim rectangles
`Quadri2d`	2-dim quadrilaterals
`Polygon`	(2-dim) polygons
`Tetra`	Tetrahedra
`Brick`	3-dim rectangles
`Hexa`	Hexahedra (faces maybe not be flat)
`Prism`	Cartesian product of a 2-dim cell with an interval

Equipped with *shape functions*, these cells become *finite elements* (classes)

C++ Polymorphism for WG on Polytopal Meshes

- WGFEM can be conveniently used on a polytopal mesh with elements of different geometric shapes,
  e.g., triangles, quadrilaterals, and more general polygons
- Implementation of WG elements can be unified for
  – triangles, rectangles, quadrilaterals, ...
  via C++ polymorphism and `template`
  (instantiation "on the fly")

WG C++ Libraries: Integration with Other Packages

Mesh generation

✓	PolyMesher	Matlab code (thru Flat File Transfer (FFT))
✓	TetGen	A tetrahedral mesh generator (FFT, Linking)
...	CUBIT/Trelis	A hexahedral mesh generator

Linear Solvers: PETSc

Visualization

✓	Silo	A mesh & field I/O library, scientific database
...	VisIt	An interactive visualization tool (interactive computing!)

? FreeFEM++: Use its script language

Case Study: Solving Darcy Equation

By lowest order WG
– No need for stabilization
– Minimum DOFs

Recall: The Darcy flow problem is usually formulated as

$$\begin{cases} \nabla \cdot (-\mathbf{K}\nabla p) \equiv \nabla \cdot \mathbf{u} = f, & \mathbf{x} \in \Omega, \\ p = p_D, & \mathbf{x} \in \Gamma^D, \\ \mathbf{u} \cdot \mathbf{n} = u_N, & \mathbf{x} \in \Gamma^N, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a bounded polygonal domain, p the unknown pressure, $\mathbf{K}$ a permeability tensor that is uniformly symmetric positive-definite, f a source term, p_D, u_N are respectively Dirichlet and Neumann boundary data, $\mathbf{n}$ the unit outward normal vector on $\partial\Omega$, which has a nonoverlapping decomposition $\Gamma^D \cup \Gamma^N$.

Note: Here $\mathbf{K}$ is an order 2 or 3 full SPD matrix.

Scheme for Pressure
Seek $p_h = \{p_h^\circ, p_h^\partial\} \in S_h(l,m)$ such that $p_h^\partial|_{\Gamma^D} = Q_h^\partial p_D$ and

$$\mathcal{A}_h(p_h, q) = \mathcal{F}(q), \qquad \forall q = \{q^\circ, q^\partial\} \in S_h^0(l,m). \tag{5}$$

where

$$\mathcal{A}_h(p_h, q) := \sum_{E \in \mathcal{E}_h} \int_E \mathbf{K} \nabla_{w,n} p_h \cdot \nabla_{w,n} q \tag{6}$$

and

$$\mathcal{F}(q) := \sum_{E \in \mathcal{E}_h} \int_E f q^\circ - \sum_{\gamma \in \Gamma_h^N} \int_\gamma u_N q. \tag{7}$$

Velocity/Flux

Velocity: L_2-projection back into subsp. of $V(E, n)$:

$$\mathbf{u}_h = \mathbb{Q}_h(-\mathbf{K}\nabla_{w,n} p_h), \tag{8}$$

Normal flux: across edges

$$\int_{e \in \partial E} \mathbf{u}_h \cdot \mathbf{n}_e$$

WG: Bilinear and Linear Forms at the Element Level

Bilinear form (matrix): The grad-grad form

$$\int_E \mathbf{K} \nabla_{w,n} p_h \cdot \nabla_{w,n} q$$

Bilinear form (matrix): Stabilizer

$$\sum_{e \in \partial E} \langle Q^\partial p^\circ - p^\partial, Q^\partial q^\circ - q^\partial \rangle$$

Linear form (vector): Source term, Boundary term

$$\int_E f q^\circ, \qquad \int_\gamma u_N q^\partial$$

Recall the 3 traditional FE types

Numer. Ex. 1

Example 1
2-dim, $\Omega = (0,1)^2$,
$p(x,y) = \sin(\pi x)\sin(\pi y)$,
Homo. Dirichlet cond. on $\partial\Omega$

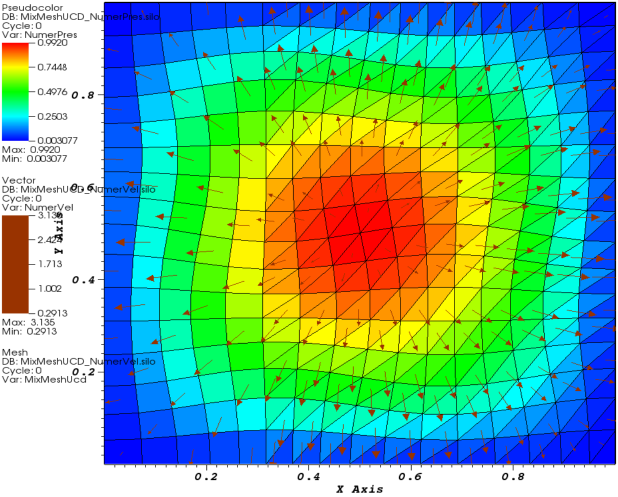
Mesh
– All triangles
– All rectangles
– All quadrilaterals
– Mix of triangles and quadrilaterals

Perturbation to rect. mesh is controlled by two parameters mx, δ:

$$\tilde{x} = \delta \sin(3\pi x)\cos(3\pi y);$$
$$\tilde{y} = -\delta \cos(3\pi x)\sin(3\pi y)$$

Numer. Ex.1: Mesh, Pressure, Velocity

$n = 16$, Left 1/3rd quadri., Perturbation $\delta = 0.02$; `Silo`, `VisIt`



1st order convergence in numerical pressure and (edge) normal flux

Mix of triangles and quadrilaterals:
Left 1/3rd quadrilaterals
Perturbation parameter $\delta = 0.02$

1/h	$\|p - p_h^\circ\|_{L_2(\Omega)}$	$\max_{\gamma\in\Gamma_h} \|\mathbf{u}\cdot\mathbf{n}_\gamma - \mathbf{u}_h\cdot\mathbf{n}_\gamma\|_{L_2(\gamma)}$	# iterations (CG)
16	3.530e-2	5.754e-1	245
32	1.766e-2	3.025e-1	503
64	8.863e-3	1.598e-1	958
128	4.432e-3	8.057e-2	1916
rate	1st order	1st order	1st order

Numer. Ex.1 (cont'd)

1st order convergence in numerical pressure and (edge) normal flux

All quadrilaterals: perturbation parameter $\delta = 0.05$

1/h	$\|p - p_h^\circ\|_{L_2(\Omega)}$	$\max_{\gamma\in\Gamma_h} \|\mathbf{u}\cdot\mathbf{n}_\gamma - \mathbf{u}_h\cdot\mathbf{n}_\gamma\|_{L_2(\gamma)}$
16	4.212e-2	8.656e-1
32	2.114e-2	4.414e-1
64	1.058e-2	2.303e-1
128	5.293e-3	1.169e-1
rate	1st order	1st order

Notes:

- Elevated interstitial fluid pressure reduces drug efficacy;
  Welter,Rieger, *PLoS ONE* (2013)
  FDA always looks at drug efficacy and toxicity.
- Accurate normal flux is needed for FVM for transport prob.
  Ginting,Lin,Liu, *J.Sci.Comput.* (2015)

A Bit of Theory

Accuracy of the lowest order WGFEM

Assumptions

- $p \in H^2$
- Mesh: *Asymptotically parallelogram*

Proposition. For the elliptic problem and WG scheme with the lowest order elements on mix meshes, there holds 1st order convergence in the numerical pressure and normal flux:

$$\|p - p_h^\circ\|_{L_2(\Omega)} = \mathcal{O}(h),$$

and

$$\max_{\gamma \in \Gamma_h} \|\mathbf{u} \cdot \mathbf{n}_\gamma - \mathbf{u}_h \cdot \mathbf{n}_\gamma\|_{L_2(\gamma)} = \mathcal{O}(h).$$

Solving Darcy. Eqn. by WG (P_0, P_0, RT_0) on Tetra. Mesh

Calling of Libraries

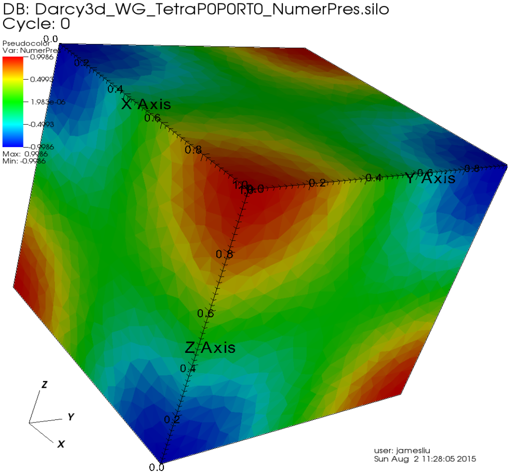
- Tetrahedral meshes generated by `TetGen`;
- WG FE scheme for numerical pressure;
- A SPD lin. sys. solved by a Conjugate Gradient solver in `LinLite`, which will be replaced by / cast onto `PETSc`;
- Numerical pressure plotted by `VisIt`.

A particular example on $[0,1]^3$:

– Known exact pressure $p(x,y,z) = \cos(\pi x)\cos(\pi y)\cos(\pi z)$

– Simple permeability $\mathbf{K} = \mathbb{I}_3$

Numer. Ex.2: WG (P_0, P_0, RT_0) Tetra.



Numer. Ex.2: WG (P_0, P_0, RT_0) Tetra.

Results are as expected

Table: Convergence rates of errors

1/h	$\|p - p_h^\circ\|_{L_2(\Omega)}$	$\|\|\|p - p_h^\circ\|\|\|_h$
8	4.883e-2	4.817e-3
16	2.451e-2	1.202e-3
32	1.226e-2	2.853e-4
	1st order	2nd order

where the discrete L_2 error in pressure is defined as

$$|||p - p_h|||_h^2 = \sum_{E \in \mathcal{E}_h} (p - p_h)^2(E_c)|E|$$

and E_c is the element center.

FEMs for Darcy/Elliptic Eqn. on Hexa. Meshes

Wheeler,Xue,Yotov, *Numer.Math.* (2012)
MFEM: Enhanced $BDDF_1$ elements and special quadrature used

Falk,Gatto,Monk, *M2AN* (2011)
$H(\text{div})$, $H(\text{curl})$-conforming FE subspace on hexa.

- General trilin. mapping of $RT_{[0]}$ does not contain const. vec.;
- To have const. vec., need a dim-21 subsp. of $RT_{[1]}$ (dim=36).

Zhang, *JCAM* (2007); Zhang, *Numer.Math.* (2004)
Nested refinements, Quadratures

Naff,Russell,Wilson, *Comput. Geosci.* (2002)
Need primary & secondary (e.g., $\hat{x} = \frac{1}{2}$) faces to be flat

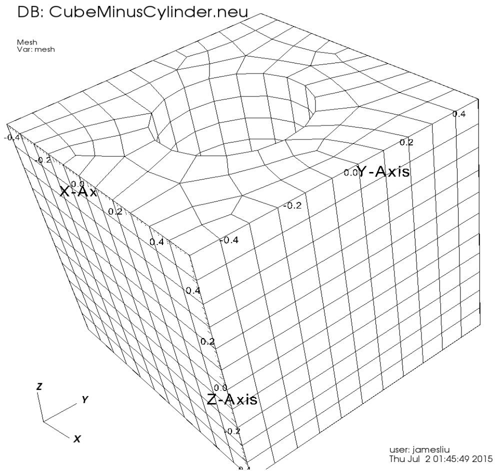
Why Hexahedral Meshes

- For complicated domains, e.g., artery (applications in drug delivery)
- Permeabilities in radial, angular, vertical directions are quite different:

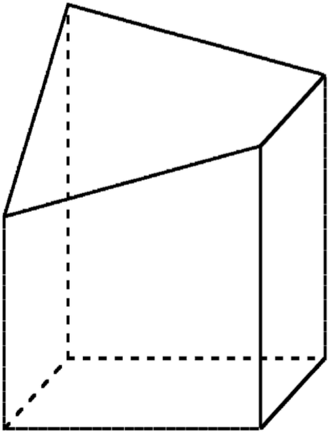
$$\mathbf{K} = Q^T \mathbf{K}_c Q, \quad \mathbf{K}_c = \mathrm{diag}(K_r, K_\theta, K_z), \quad Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Cylindrical coordinates might better characterize the problems.
- **Choices**:
  – Isogeometric FEs; See Hossain et al., *Comput. Mech.* (2012)
  – Cartesian FEs on hexahedral meshes

CUBIT/Trelis: Hexahedral Mesh Generator



A General Hexahedron: Non-flat Face(s)

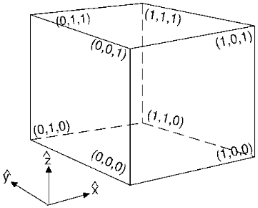


Hexahedron: Trilinear Mapping from the unit cube $[0,1]^3$

$$\mathbf{v} = \mathbf{v}_{000} + \mathbf{v}_a\hat{x} + \mathbf{v}_b\hat{y} + \mathbf{v}_c\hat{z} + \mathbf{v}_d\hat{y}\hat{z} + \mathbf{v}_e\hat{z}\hat{x} + \mathbf{v}_f\hat{x}\hat{y} + \mathbf{v}_g\hat{x}\hat{y}\hat{z}.$$

$$\begin{aligned}
\mathbf{v}_a &= \mathbf{v}_{100} - \mathbf{v}_{000}, & \mathbf{v}_d &= (\mathbf{v}_{011} - \mathbf{v}_{000}) - (\mathbf{v}_b + \mathbf{v}_c),\\
\mathbf{v}_b &= \mathbf{v}_{010} - \mathbf{v}_{000}, & \mathbf{v}_e &= (\mathbf{v}_{101} - \mathbf{v}_{000}) - (\mathbf{v}_c + \mathbf{v}_a),\\
\mathbf{v}_c &= \mathbf{v}_{001} - \mathbf{v}_{000}, & \mathbf{v}_f &= (\mathbf{v}_{110} - \mathbf{v}_{000}) - (\mathbf{v}_a + \mathbf{v}_b),
\end{aligned}$$
$$\mathbf{v}_g = (\mathbf{v}_{111} - \mathbf{v}_{000}) - ((\mathbf{v}_a + \mathbf{v}_b + \mathbf{v}_c) + (\mathbf{v}_d + \mathbf{v}_e + \mathbf{v}_f)).$$

Fact: A hexahedron becomes parallelepiped iff $\mathbf{v}_g = \mathbf{0}$.



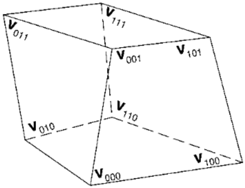


Figure: Naff,Russell,Wilson, *Comput.Geosci.* (2002)

Hexahedron: Jacobian

Recall trilinear mapping $F : [0,1]^3 \longrightarrow E$ (hexahedron)

$$\mathbf{v} = \mathbf{v}_{000} + \mathbf{v}_a \hat{x} + \mathbf{v}_b \hat{y} + \mathbf{v}_c \hat{z} + \mathbf{v}_d \hat{y}\hat{z} + \mathbf{v}_e \hat{z}\hat{x} + \mathbf{v}_f \hat{x}\hat{y} + \mathbf{v}_g \hat{x}\hat{y}\hat{z}.$$

Tangential vectors

$$\frac{\partial \mathbf{v}}{\partial \hat{x}} =: \mathbf{X}(\hat{y}, \hat{z}), \qquad \frac{\partial \mathbf{v}}{\partial \hat{y}} =: \mathbf{Y}(\hat{z}, \hat{x}), \qquad \frac{\partial \mathbf{v}}{\partial \hat{z}} =: \mathbf{Z}(\hat{x}, \hat{y}).$$

Jacobian matrix $\mathbf{J}_F$

Jacobian determinant $J_F = \Big(\mathbf{X}(\hat{y}, \hat{z}) \times \mathbf{Y}(\hat{z}, \hat{x})\Big) \cdot \mathbf{Z}(\hat{x}, \hat{y})$

Hexahedron: Piola Transformation

Piola transformation maps
a vector field $\hat{\mathbf{w}}(\hat{x}, \hat{y}, \hat{z})$ on the unit cube $\hat{E}$ to
a vector field $\mathbf{w}(x, y, z)$ on a hexahedron E:

$$\mathbf{w}(x, y, z) := \frac{\mathbf{J}_F}{J_F} \hat{\mathbf{w}}(\hat{x}, \hat{y}, \hat{z})$$

Benefits: Piola transformation preserves the normal flux on each (boundary) face:

$$\int_{\partial \hat{E}} \hat{\mathbf{w}} \cdot \hat{\mathbf{n}} = \int_{\partial E} \mathbf{w} \cdot \mathbf{n}$$

WGFEMs for Darcy on Hexa. Meshes: Main Ideas

- Solving for primal variable (pressure)
- Approximate ∇p by discrete weak gradient $\nabla_{w,n} p_h$
- No use of Piola transformation
- Hierarchy of approximations:
  from $(Q_0, Q_0, RT_{[0]})$ to (Q_0, Q_1, P_1^3) to higher order

Theorem 1 (Local Mass Conservation)
Let E be any hexahedral element. There holds

$$\int_E f - \int_{\partial E} \mathbf{u}_h \cdot \mathbf{n} = 0. \tag{9}$$

Proof. In Equation (5), take a test function q such that $q|_{E^\circ} = 1$ but vanishes everywhere else. We thus obtain

$$\begin{aligned}
\int_E f &= \int_E (\mathbf{K}\nabla_{w,n} p_h) \cdot \nabla_{w,n} q \\
&= \int_E \mathbb{Q}_h(\mathbf{K}\nabla_{w,n} p_h) \cdot \nabla_{w,n} q \\
&= -\int_E \nabla \cdot \mathbb{Q}_h(\mathbf{K}\nabla_{w,n} p_h) \\
&= -\int_{\partial E} \mathbb{Q}_h(\mathbf{K}\nabla_{w,n} p_h) \cdot \mathbf{n} \\
&= \int_{\partial E} \mathbf{u}_h \cdot \mathbf{n}
\end{aligned}$$

Theorem 2 (Normal Flux Continuity)
Let γ be a (nonplanar) face shared by two hexahedra E_1, E_2 and $\mathbf{n}_1, \mathbf{n}_2$ be respectively the (varying) outward unit normal vector (of E_1, E_2. There holds

$$\int_\gamma \mathbf{u}_h^{(1)} \cdot \mathbf{n}_1 + \int_\gamma \mathbf{u}_h^{(2)} \cdot \mathbf{n}_2 = 0. \tag{10}$$

Proof. In Equation (5), take a test function $q = (q^\circ, q^\partial)$ such that
– $q^\circ \equiv 0$ on interior of all hexahedra;
– q^∂ nonzero on γ;
– $q^\partial = 0$ on all faces other than γ.
Obtain

$$\int_\gamma \left(\mathbf{u}_h^{(1)} \cdot \mathbf{n}_1 + \mathbf{u}_h^{(2)} \cdot \mathbf{n}_2 \right) q^\partial = 0.$$

Proposition. For $(Q_0, Q_0, RT_{[0]})$:
Let p be the exact pressure solution of the Darcy BVP and p_h be WG numerical pressure from Scheme (5) with $(Q_0, Q_0, RT_{[0]})$.
There holds

$$\|p - p_h\|_{L_2(\Omega)} \leq Ch. \tag{11}$$

Example 3:
$\Omega = [0,1]^3$, $\mathbf{K} = \mathbb{I}_3$, known exact pressure

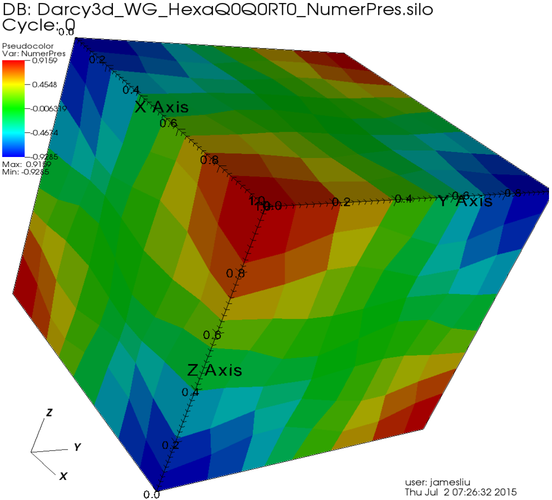
$$p(x,y,z) = \cos(\pi x)\cos(\pi z)\cos(\pi z).$$

Uniform brick mesh perturbed with $\delta = 1$

$$\begin{cases} x = \hat{x} + \delta * 0.03 \sin(3\pi\hat{x})\cos(3\pi\hat{y})\cos(3\pi\hat{z}) \\ y = \hat{y} - \delta * 0.04 \cos(3\pi\hat{x})\sin(3\pi\hat{y})\cos(3\pi\hat{z}) \\ z = \hat{z} + \delta * 0.05 \cos(3\pi\hat{x})\cos(3\pi\hat{y})\sin(3\pi\hat{z}) \end{cases}$$

See also Wheeler,Xue,Yotov, *Numer.Math.* (2012)

Numer. Ex.3: WG Hexa. $(Q_0, Q_0, RT_{[0]})$ Numer. Pres.



Numer. Ex.3: WG Hexa. $(Q_0, Q_0, RT_{[0]})$: Error

Table: Convergence rates of errors

1/h	$\|p - p_h\|_{L_2(\Omega)}$	$\|\|\|p - p_h\|\|\|_h$	Flux discrepancy
8	7.001e-2	8.042e-3	1.748e-15
16	3.562e-2	2.108e-3	2.367e-15
32	1.790e-2	5.358e-4	3.939e-15
64	8.965e-3	1.345e-4	3.380e-15
	1st order	2nd order	Double precision (zero)

where the discrete L_2 error in pressure is defined as

$$|||p - p_h|||_h^2 = \sum_{E \in \mathcal{E}_h} (p - p_h)^2 (E_c) |E|$$

and E_c is the element center.

Questions

(i) Dimension-independent implementation?
Unified treatment of 2-dim & 3-dim elements
(like `deal.II`)

(ii) Unified treatment of simplicial and tensor-type elements?
Simplicial: triangle, tetrahedron
Tensor-type: rectangle (2d), brick (3d)
More general: quadrilateral, hexahedral, ...

(iii) Unified treatment of scalar/vector-valued WG elements?
Weak gradient/div/curl

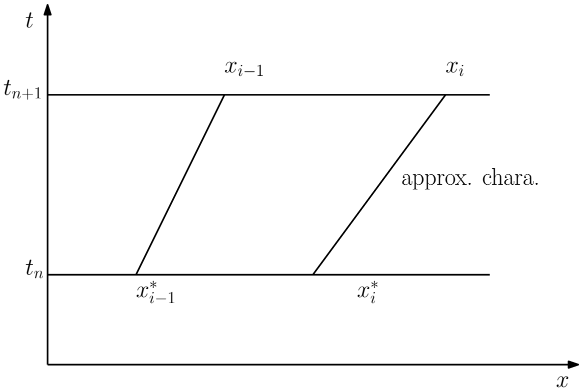
- ...

Further Applications of WG FEMs

- WG for time-dependent conviction-diffusion (transport) eqn.
  Space-time WG finite elements?
  Spatial WG finite elements + characteristic tracking ?
- WG applied to drug transport problems
  - Darcy equation
  - Stokes equation
  - Transport equation
  - Two-phase problems
    - WG (Darcy eqn.) + FVM (transport)
      Ginting,Lin,Liu, *JSC* (2015)
    - ?

WG Finite Elements on Space-time Polygonal Stripes

WG + chara. tracking



Thanks to Lin Mu !