Polygonal Splines and Their Applications for Numerical Solution of PDE [1]

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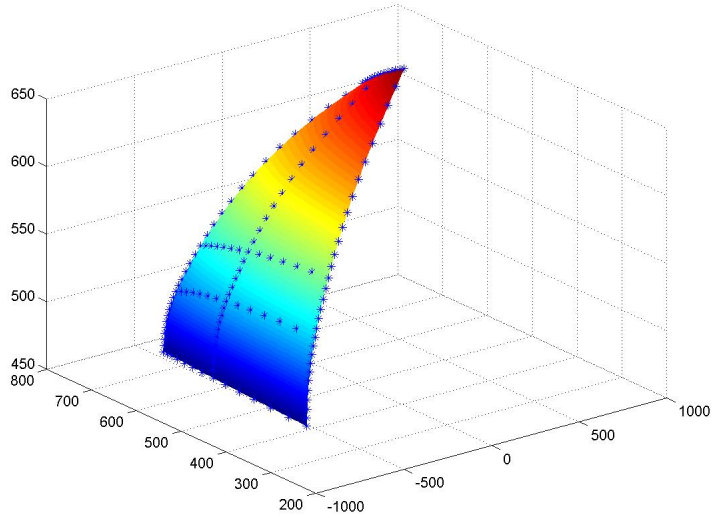
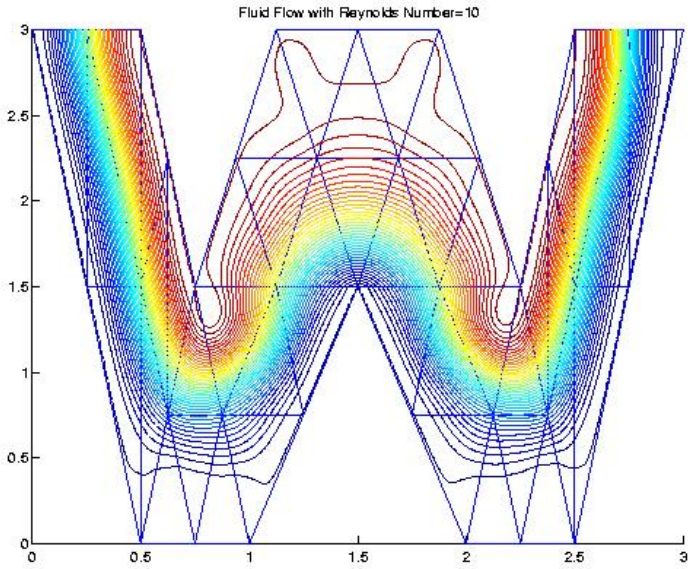
Introduction

Multivariate splines, usually defined on a triangulation in 2D, or a tetrahedral partition in 3D, or spherical surface, or a simplicial partition in $\mathbb{R}^n$, have been developed for 30 years and they are extremely useful to various numerical applications such as computer aided geometric design, numerical solutions of various linear and nonlinear partial differential equations, scattered data interpolation and fitting, image enhancements, spatial statistical analysis and data forecasting, and etc..





Triangulated Splines for Applications

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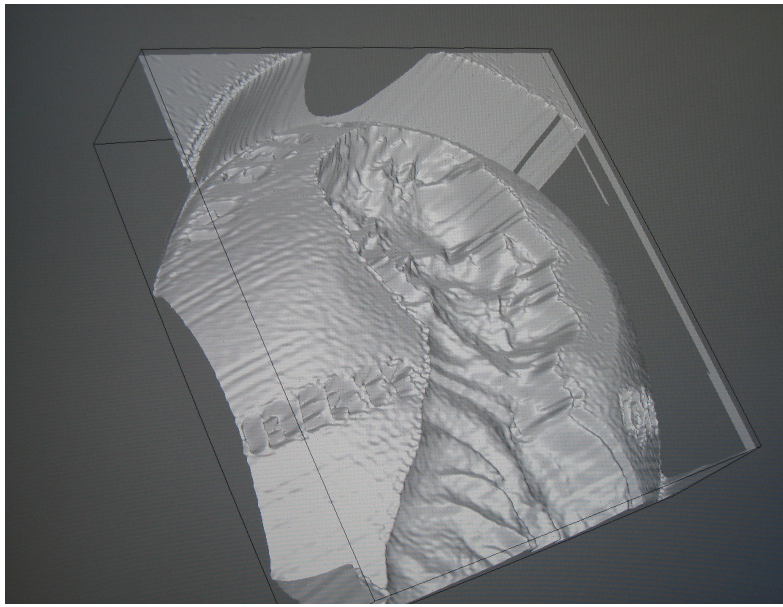
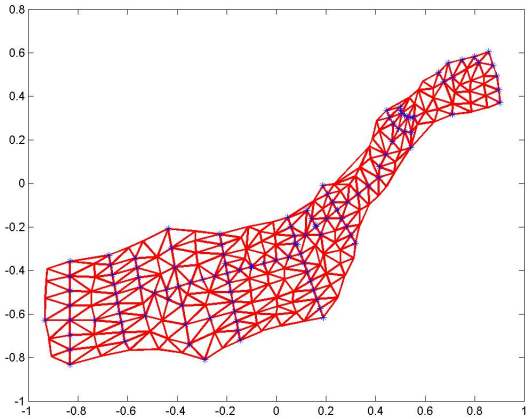
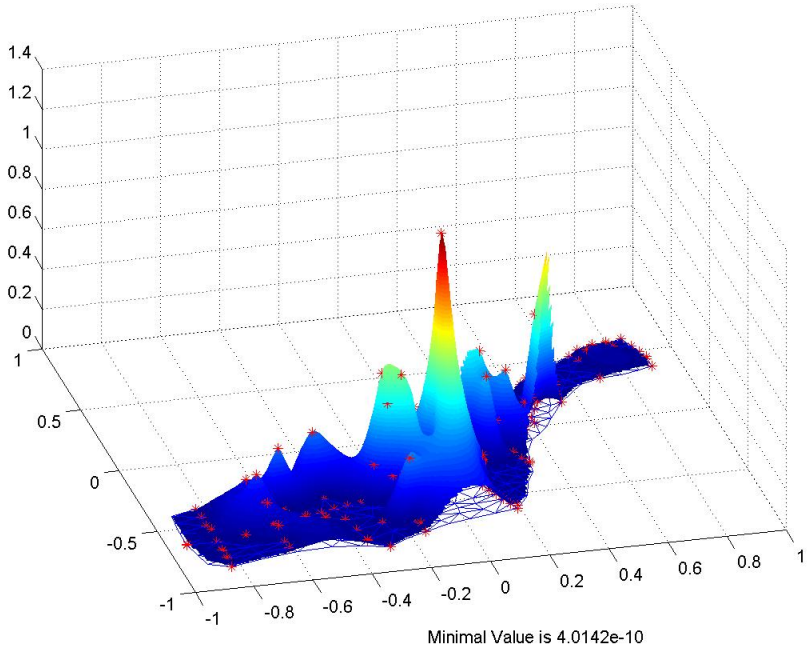


2D spline interpolation[2]

[2] M. -J. Lai and L. L. Schumaker, Domain decomposition method for scattered data fitting, SIAM J. Num. Anal. 47(2009) 911–928.

Triangulated Splines for Applications (II)





[3]

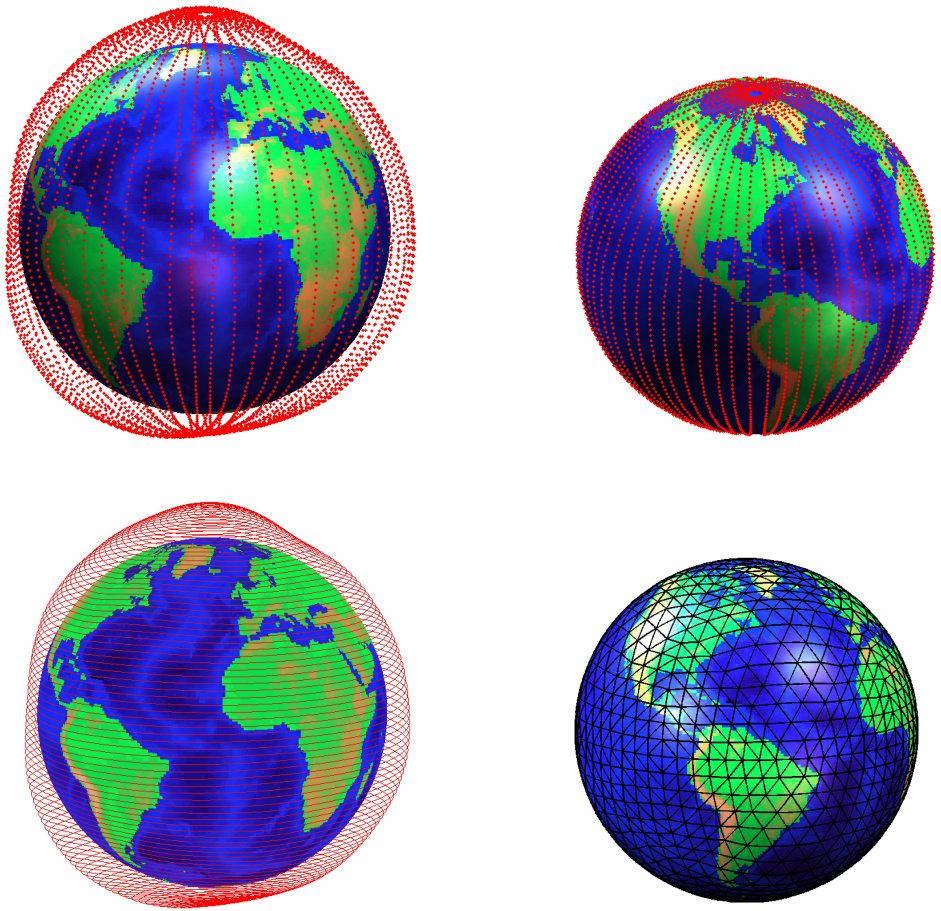
[3] M. -J. Lai and Meile, C. , Scattered data interpolation with nonnegative preservation using bivariate splines, Computer Aided Geometric Design, vol. 34 (2015) pp. 37–49

Triangulated Splines for Applications (III)

[4]

[4] Guo, W. H. and Lai, M. -J., Box Spline Wavelet Frames for Image Edge Analysis, SIAM Journal Imaging Sciences, vol. 6 (2013) pp. 1553–1578.

Triangulated Splines for Applications (IV)



~~See~~ [5]

[5] Lai, M. J., Shum, C. K., Baramidze, V. and Wenston, P., Triangulated Spherical Splines for Geopotential Reconstruction, Journal of Geodesy, vol. 83 (2009) pp. 695–708.

What Are Multivariate Splines?

Let Δ be a triangulation of a domain $\Omega \subset \mathbb{R}^2$. For integers $d \geq 1$, $-1 \leq r \leq d$ define by

$$S_d^r(\Delta) = \{s \in C^r(\Omega), s|_t \in \mathbf{P}_d, t \in \Delta\}$$

the spline space of smoothness r and degree d over Δ.

In general, let $\mathbf{r} = (r_1, \cdots, r_n)$ with $r_i \geq 0$ be a vector of integers. Define

$$S_d^{\mathbf{r}}(\Delta) = \{s \in C^{-1}(\Omega), s|_{e_i} \in C^{r_i}, e_i \in E\},$$

where E is the collection of interior edges of $\triangle$. Each spline in $S_d^{\mathbf{r}}(\triangle)$ has variable smoothness.
This can handle the situation of hanging nodes in a triangulation!

Definition of Spline Functions

Let $T = \langle (x_1, y_1), (x_2, y_2), (x_3, y_3) \rangle$. For any point (x, y), let b_1, b_2, b_3 be the solution of

$$\begin{aligned} x &= b_1 x_1 + b_2 x_2 + b_3 x_3 \\ y &= b_1 y_1 + b_2 y_2 + b_3 y_3 \\ 1 &= b_1 + b_2 + b_3. \end{aligned}$$

Fix a degree $d > 0$. For $i + j + k = d$, let

$$B_{ijk}(x, y) = \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k$$

which is called Bernstein-Bézier polynomials.
For each $T \in \Delta$, let

$$S|_T = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}(x, y).$$

We use $\mathrm{s} = (c_{ijk}^T, i + j + k = d, T \in \Delta)$ be the coefficient vector to denote a spline function in $S_d^{-1}(\Delta)$.

Evaluation and Derivatives

We use the de Casteljau algorithm to evaluate a Bernstein-Bézier polynomial at any point inside the triangle. It is a simple and stable computation.
Let $T = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ and $S|_T = \sum_{i+j+k=d} c_{ijk} B_{ijk}(x, y)$. Then directional derivative

$$D_{\mathbf{v}_2 - \mathbf{v}_1} S|_T = d \sum_{i+j+k=d-1} (c_{i,j+1,k} - c_{i+1,j,k}) B_{ijk}(x, y).$$

Similar for $D_{\mathbf{v}_3 - \mathbf{v}_1} S|_T$.
D_x and D_y are linearly combinations of these two directional derivatives.

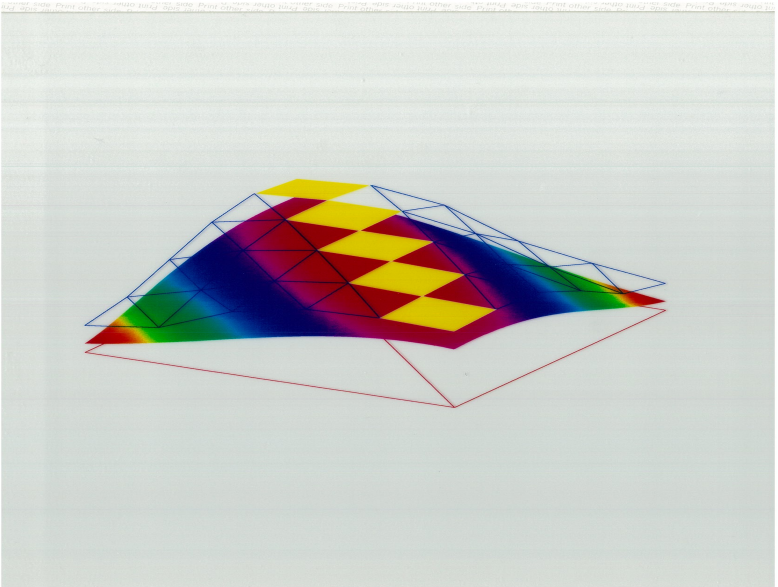
Smoothness Condition between Triangles

Let T_1 and T_2 be two triangles in Δ which share a common edge e. Then $S \in C^r(T_1 \cup T_2)$ if and only if the coefficients of $c_{ijk}^{T_1}$ and $c_{ijk}^{T_2}$ satisfy the following linear conditions. E.g.,

$$S \in C^0(T_1 \cup T_2) \text{ iff } c_{0,j,k}^{T_1} = c_{j,k,0}^{T_2}, j + k = d$$

$$S \in C^1(T_1 \cup T_2) \text{ iff } c_{1,j,k}^{T_1} = b_1 c_{j+1,k,0}^{T_2} + b_2 c_{j,k+1,0}^{T_2} + b_3 c_{j,k,1}^{T_2}$$

for $i + k = d - 1$ and etc. (cf. [Farin, 86] and [de Boor, 87]). We code them by H**c**=0.



Integration

Let s be a spline in $S_d^r(\triangle)$ with $s|_T = \sum_{i+j+k=d} c_{ijk}^T B_ijk(x,y), T \in \triangle$. Then

$$\int_\Omega s(x,y)dxdy = \sum_{T\in\triangle} \frac{A_T}{\binom{d+2}{2}} \sum_{i+j+k=d} c_{ijk}^T.$$

If $p = \sum_{i+j+k=d} a_{ijk} B_{ijk}(x,y)$ and $q = \sum_{i+j+k=d} b_{ijk} B_{ijk}(x,y)$ over a triangle T, then

$$\int_T p(x,y)q(x,y)dxdy = \mathbf{a}^\top M_d \mathbf{b},$$

where $\mathbf{a} = (a_{ijk}, i+j+k=d)^\top$, $\mathbf{b} = (b_{ijk}, i+j+k=d)^\top$, M_d is a symmetric matrix with known entries (cf. [Chui and Lai, 1992]). Similarly, we have (cf. [Awanou and Lai, 2005])

$$\int_T p(x,y)q(x,y)r(x,y)dxdy = \mathbf{a}^\top A_d \mathbf{b} \odot \mathbf{c}.$$

Spline Approximation Order

We have (cf. [Lai and Schumaker'98][6])

Theorem

Suppose that $\triangle$ is a β-quasi-uniform triangulation of domain $\Omega \in \mathbf{R}^2$ and suppose that $d \geq 3r+2$. Fix $0 \leq m \leq d$. Then for any f in a Sobolev space $W_p^{m+1}(\Omega)$, there exists a quasi-interpolatory spline $Q_f \in S_d^r(\triangle)$ such that

$$\|f - Q_f\|_{k,p,\Omega} \leq C|\triangle|^{m+1-k}|f|_{d+1,p,\Omega}, \forall 0 \leq k \leq m+1,$$

for a constant $C > 0$ independent of f, but dependent on β and d.

See more detail in monograph[7]

[6] Lai, M. J. and Schumaker, L. L., Approximation Power of Bivariate Splines, Advances in Computational Mathematics, vol. 9 (1998) pp. 251–279.

[7] M. -J. Lai and L. L. Schumaker, Spline Functions on Triangulations, Cambridge University Press, 2007.

The Weak Form of PDE's

The weak formulation for elliptic PDE's reads : find $u \in H^k(\Omega)$ which satisfies its boundary condition such that

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in H_0^r(\Omega), \tag{1}$$

where $a(u,v)$ is the bilinear form defined by

$$a(u,v) = \begin{cases} \int_\Omega \nabla u \cdot \nabla v dxdy, & k=1; \\ \int_\Omega \triangle u \triangle v dxdy, & k=2, \end{cases}$$

and $\langle f, v\rangle = \int_\Omega f(x,y)v(x,y)dxdy$ is the L_2 inner product of f and v. Here $H^k(\Omega)$ and $H_0^k(\Omega)$ are standard Sobolev spaces. Clearly, our model problem is the Euler-Lagrange equation associated with a minimization prpblem:

$$\min_{u \in H^k(\Omega)} E(u), \; u \text{ satisfies given boundary conditions} \tag{2}$$

where the energy functional $E(u) = \frac{1}{2}a(u,u) - \langle f, u\rangle$.

Our Spline Method

[1] We write $s \in S_d^{-1}(\triangle)$ in

$$s(x,y)|_t = \sum_{i+j+k=d_t} c^t_{i,j,k} B^t_{i,j,k}(x,y), \quad (x,y) \in t \in \triangle.$$

Let $\mathbf{c} = (c^t_{i,j,k}, i+j+k = d_t, t \in \triangle)$ be the B-coefficient vector associated with s.
[2] We compute mass and stiffness matrices:

$$E(s) = \frac{1}{2}\mathbf{c}^\top K \mathbf{c} - \mathbf{c}^\top M \mathbf{f}$$

where K and M are diagonally block matrices.
[3] Since $s \in S_d^{\mathbf{r}}(\triangle)$, we have the smoothness conditions $H\mathbf{c} = 0$. Also, the boundary condition can be written as $B\mathbf{c} = \mathbf{g}$.
[4] FInally we solve the constrained minimization problem:

$$\min E(s), \text{ subject to } H\mathbf{c} = 0,\ B\mathbf{c} = \mathbf{g}.$$

Our Spline Method (II)

By using Lagrange multiplier method, we let

$$L(\mathbf{c},\alpha,\beta) = \frac{1}{2}\mathbf{c}^T K\mathbf{c} - \mathbf{c}^T\mathbf{M}\mathbf{f} + \alpha H\mathbf{c} + \beta(B\mathbf{c} - \mathbf{g})$$

and compute local minimizers. That is, we solve

$$\begin{bmatrix} B^T & H^T & K \\ 0 & 0 & B \\ 0 & 0 & H \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} M\mathbf{f} \\ \mathbf{g} \\ 0 \end{bmatrix} \tag{3}$$

which can be solved using the least squares method or by an iterative algorithm in [Awanou, Lai, Wenston'06[8]].

We minimize
$\widetilde{L}(\mathbf{c},\alpha,\beta) = \frac{1}{2}\mathbf{c}^T K\mathbf{c} - \mathbf{c}^T M\mathbf{f} + \alpha\|H\mathbf{c}\|^2 + \beta\|B\mathbf{c} - \mathbf{g}\|^2.$

[8] Awanou, G., Lai, M. J. and Wenston, P., The Multivariate Spline Method for Scattered Data Fitting and Numerical Solution of Partial Differential Equations, Wavelets and Splines, Nashboro Press, (2006) edited by G. Chen and Lai, M. J. pp. 24–74

An Iterative Algorithm

We begin with

$$\left(K + \frac{1}{\epsilon}[B^\top, H^\top]^\top [B^\top H^\top]\right) \mathbf{c}^{(1)} = M\mathbf{f} + \frac{1}{\epsilon}[B^\top, H^\top]^\top G$$

Then we do the following iterative step

$$\left(K + \frac{1}{\epsilon}[B^\top, H^\top]^\top [B^\top H^\top]\right) \mathbf{c}^{(k+1)} = K\mathbf{c}^{(k)} + \frac{1}{\epsilon}[B^\top, H^\top]^\top G$$

for $k = 1, 2, \cdots, 10$ and $\epsilon > 0$, e.g., $\epsilon = 10^{-6}$ with $\mathbf{c} = 0$.

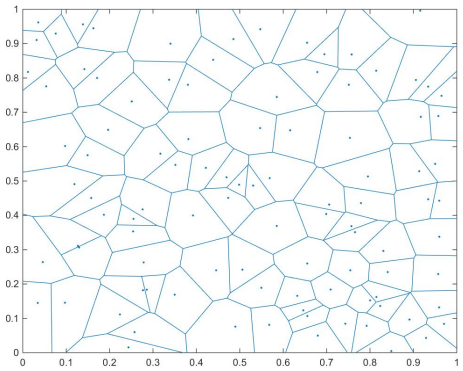
Theorem (Awanou, Lai, and Wenston'06)

Suppose that K is positive definite with respect to $[B^\top H^\top]$. Then the above iterative algorithm converges and

$$\|\mathbf{c}^{(k+1)} - \mathbf{c}\| \le C\epsilon^k, \quad \forall k \ge 1.$$

A Natural Question

A natural question follows: can we define splines on a partition containing not only triangles, but also other polygons, say over a Voronoi diagram (i.e. Dirichlet tessellation) or a patio?

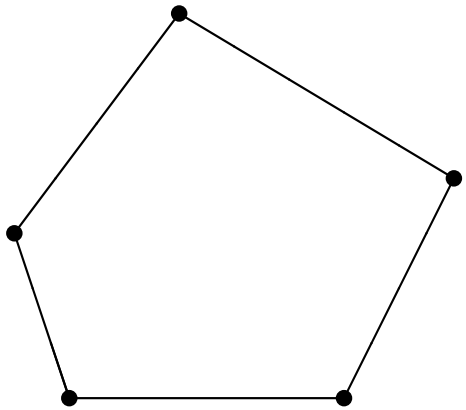


That is, can we be more versatile in solution of PDE? Can we be more efficient than the standard finite element method?

These questions have been raised in the community of numerical solution of PDE. The so-called virtual element method and the discontinuous Galerkin method are developed recently.

Generalized Barycentric Coordinates

Similar to the barycentric coordinates b_1, b_2, b_3 associated with a $T = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$, on an arbitrary polygon $P = \langle \mathbf{v}_1, \cdots, \mathbf{v}_n \rangle$, say pentagon



there are functions $b_1 \geq 0, \cdots, b_n \geq 0$ satisfying

$$\sum_{i=1}^{n} b_i(\mathbf{x}) = 1 \text{ and } \mathbf{x} = \sum_{i=1}^{n} b_i(\mathbf{x})\mathbf{v}_i \quad \forall \mathbf{x} \in \mathbb{R}^2 \tag{4}$$

which are called generalized barycentric coordinates (GBC).

GBC functions

There are many kinds of GBC available. Construction of various GBC is pioneered by Wachspress in 1975. More and more GBC have been invented.

- Wachspress Coordinates,
- Mean Value Coordinates,
- Bilinear Coordinates,
- maximum entropy coordinates,
- discrete harmonic coordinates,
- Wachspress and mean value coordinates have 3D generalization and been extended in the spherical setting.

We refer to [Floater, 2015[9]] and [G. Manzini, A. Russo and N. Sukumar 2014[10]] for a summary surveying various types of GBC.

[9] M. Floater, Generalized barycentric coordinates and applications, Acta Numerica 24 (2015), 161–214.

[10] G. Manzini, A, Russo and N. Sukumar, New perspectives on polygonal and polyhedral finite element methods, Mathematical Models and Methods in Applied Sciences Vol. 24, No. 8 (2014) 1665–1699.

Wachspress Coordinates

Example (Wachspress coordinates)

For any polygon P_n with n sides, let

$$A_i(\mathbf{x}) = A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}), \quad C_i = A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1})$$

be the signed area of triangles $\langle \mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1} \rangle$ and $\langle \mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1} \rangle$, respectively. Setting $w_i(\mathbf{x}) = C_i/(A_{i-1}(\mathbf{x})A_i(\mathbf{x}))$ we define

$$\phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^n w_j(\mathbf{x})}, \quad i = 1, 2, \cdots, n \tag{5}$$

which are called Wachspress barycentric coordinates. One can check these ϕ_i satisfy (4).
These coordinates have been developed in [Wachspress, 1975], [Warren, 1996], ... The above form was given in [Meyer et al, 2002[11]]

[11] M. Meyer, A. Barr, H. Lee and M. Desbrun (2002), Generalized barycentric coordinates for irregular polygons, J. Graph. Tools 7, 13–22.

Polygonal Bernstein-Bézier Functions

For each n-gon P_n and for any GBC $b_1, \cdots, b_n$, we let

$$B_{\mathbf{j}}^d(\mathbf{x}) = \frac{d!}{\mathbf{j}!} \prod_{i=1}^{n} b_i^{j_i}, \quad |\mathbf{j}| = d \tag{6}$$

be the polygonal Bernstein-Bézier Functions, where $\mathbf{j} = (j_1, \cdots, j_n), j_i, i = 1, \cdots, n$ are nonnegative integers, $\mathbf{j}! = \prod_{i=1}^{n} j_i!$ and $|\mathbf{j}| = j_1 + \cdots + j_n$.
Let $\Phi_{n,d}$ be the linear space of functions of the form

$$s(\mathbf{x}) = \sum_{|\mathbf{j}|=d} c_{\mathbf{j}} B_{\mathbf{j}}^d(\mathbf{x}), \qquad \mathbf{x} \in P_n \tag{7}$$

with real coefficients $c_{\mathbf{j}}$ and let

$$\mathcal{S}_d^r(\mathbf{\Delta}) = \{s \in C^r(\Omega) : s|_{P_n} \in \Phi_{n,d}, P_n \in \mathbf{\Delta}\}. \tag{8}$$

be the polygonal spline space of smoothness r and degree d, where $\mathbf{\Delta}$ is a collection of polygons.

Reduced Polygonal BB Functions

For convenience, we restrict ourselves to those BB functions in $\Phi_{n,d}(P)$ which enable us to reproduce polynomials of degree d. Let $\Psi_d(P_n)$ be the collection of all those terms in $\Phi_d(P_n)$ such that they are able to reproduce all polynomials of degree d. Any such functions in $\Psi_d(P_n)$ are called reduced Bernstein-Bézier functions.

Thus, let us identify these functions in $\Psi_d(P_n)$.

Recall that any $p \in \Pi_d$ has a unique, d-variate *blossom,* $\mathbb{P}[p]$*. For* p*, its blossom* $\mathbb{P}[p](\mathbf{x}_1, \ldots, \mathbf{x}_d)$ *with variables* $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \mathbb{R}^2$ *is uniquely defined by the three properties: (i) it is symmetric in the variables* $\mathbf{x}_1, \ldots, \mathbf{x}_d$*; (ii) it is multi-affine, i.e., affine in each variable while the others are fixed; and (iii) it has the diagonal property,*

$$\mathbb{P}[p](\underbrace{\mathbf{x}, \ldots, \mathbf{x}}_{d}) = p(\mathbf{x}).$$

Example 1

Consider $d = 2$ first. Let us first expand the $\mathbf{x}$ variable using the barycentric coordinates of $\mathbf{x}$ with respect to the vertex $\mathbf{v}_i$ and its two neighbors, i.e., with respect to $T_i := [\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}]$. That is,

$$\mathbf{x} = \sum_{j=-1}^{1} \lambda_{i,j}(\mathbf{x})\mathbf{v}_{i+j}, \qquad 1 = \sum_{j=-1}^{1} \lambda_{i,j}(\mathbf{x}). \tag{9}$$

By the properties (4) and (9), we have

$$\begin{aligned} p(\mathbf{x}) =& \mathbb{P}[p](\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{n} b_i(\mathbf{x})\mathbb{P}[p](\mathbf{v}_i, \mathbf{x}) \\ =& \sum_{i=1}^{n} b_i(\mathbf{x}) \sum_{j=-1}^{1} \lambda_{i,j}(\mathbf{x})\mathbb{P}[p](\mathbf{v}_i, \mathbf{v}_{i+j}). \end{aligned}$$

Example 1(Cont.)

After a simplification, we have

$$p(\mathbf{x}) = \sum_{i=1}^{n} b_i(\mathbf{x})\lambda_{i,0}(\mathbf{x})p(\mathbf{v}_i) + \sum_{i=1}^{n} \mathbb{P}[p](\mathbf{v}_i, \mathbf{v}_{i+j})(b_i(\mathbf{x})\lambda_{i,1}(\mathbf{x}) + b_{i+1}(\mathbf{x})\lambda_{i+1,-1}(\mathbf{x})). \quad (10)$$

Thus, we are ready to define

$$F_i(\mathbf{x}) = b_i(\mathbf{x})\lambda_{i,0}(\mathbf{x}) \text{ and } L_i = b_i(\mathbf{x})\lambda_{i,1}(\mathbf{x}) + b_{i+1}(\mathbf{x})\lambda_{i+1,-1}(\mathbf{x}). \quad (11)$$

We know that span$\{F_i, L_i, i = 1, \cdots, n\}$ is able to reproduce Π_2. Also, it is clear that each $\lambda_{i,j}$ can be expressed by using $b_k, k = 1, \cdots, n$ and hence, $F_i, L_i \in \Phi_2(P_n)$. These are reduced BB functions.

Reduced BB Functions of Order $d \geq 3$

Let

$$F_i = \lambda_{i,0}^{d-1}, F_{i,k} = \binom{d-1}{k} \phi_i \lambda_{i,1}^k \lambda_{i,0}^{d-1-k} + \binom{d-1}{k-1} \phi_{i+1} \lambda_{i,-1}^{d-k} \lambda_{i,0}^{k-1}.$$

Theorem (Floater and Lai, 2015[12])

For $d \geq 3$ and for Wachspress coordinates, the reduced BB function space $\Psi_d(P_n)$ is

$$\textit{span}\{F_i, i = 1, \ldots, n\} \oplus \textit{span}\{F_{i,k}, i = 1, \ldots, n, k = 1, \ldots, d-1\}$$
$$\oplus \frac{b}{W} \Pi_{d-3},$$

where P_n and $b(\mathbf{x}) = \prod_{j=1}^n h_j(\mathbf{x})$.

[12] M. Floater and M. -J. Lai, Polygonal slpine spaces and the numerical solution of the Poisson equation, submitted, (2015)

Polygonal Spline Space

Let $\mathbf{\Delta}$ be a collection of polygons with arbitrary number sides. Let $\Omega = \bigcup_{P \in \mathbf{\Delta}} P$ be the domain consisting of these polygons in $\mathbf{\Delta}$. We assume that the interiors of any two polygons P and $\tilde{P}$ from $\mathbf{\Delta}$ do not intersect and the intersection of P and $\tilde{P}$ is either their common edge e or common vertex v if the intersection is not empty. We can define a continuous polygonal spline space of order $d \geq 1$ over Ω by

$$\mathcal{S}_d(\mathbf{\Delta}) = \{s \in C(\Omega), s|_{P_n} \in \Psi_d(P_n), \forall P_n \in \mathbf{\Delta}\}, \tag{12}$$

where for $d = 1$, we simply let $\Psi_1(P_n) = \text{span}\{\phi_i, i = 1, \cdots, n\}$ and for $d \geq 2$, we use the reduced BB function space $\Psi_d(P_n)$ discussed above.

Dimension of Polygonal Spline Space $(d \geq 2)$

We can establish the dimension of $S_d(\mathbf{\Delta})$.

Theorem

Suppose that $S_d(\mathbf{\Delta})$ is a polygonal spline space based on Wachspress coordinates. Then The dimension of $S_d(\mathbf{\Delta})$ is

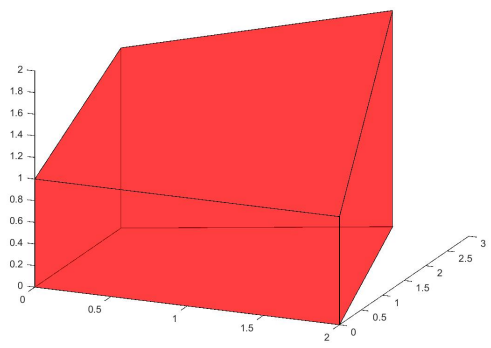
$$\dim(S_d(\mathbf{\Delta})) = \#(V) + (d-1)\#(E) + \#(\triangle)(d-1)(d-2)/2, \quad (13)$$

where V is the collection of all vertices of $\mathbf{\Delta}$, E is the collection of all edges of $\mathbf{\Delta}$ and $\#(\mathbf{\Delta})$ stands for the number of polygons in $\mathbf{\Delta}$.

Open Question: How to construct C^r smooth spline functions over a polygonal mesh $\mathbf{\Delta}$ for $r \geq 1$?

An Extension to the Trivariate Setting

For simplicity, let us consider a hexahedron $H \subset \mathbb{R}^3$. It is a polyhedron with 6 faces, 12 edges and 8 vertices.



Advantages:
(1)H is a simple polyhedron, three incidental faces at each vertex of H.
(2) One can use such hexahedrons to partition any polyhedral domain in $\mathbb{R}^3$.

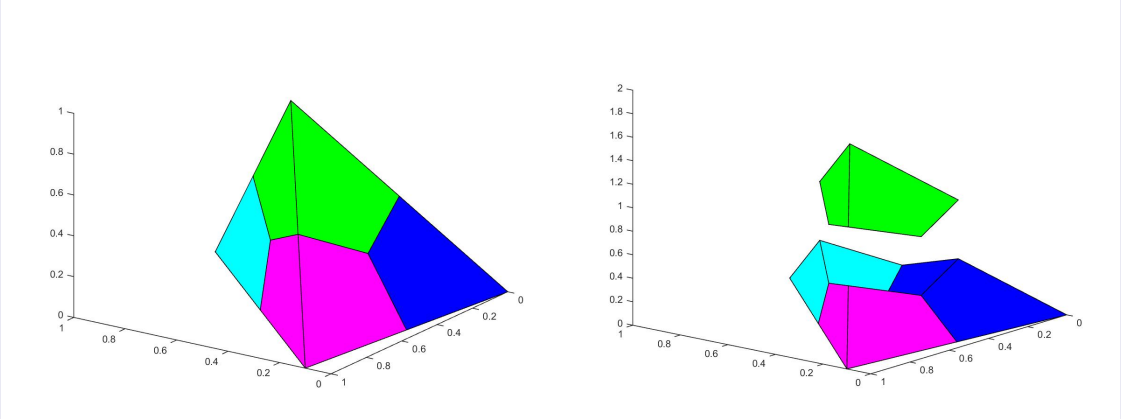
Hexahedral Partitions

Theorem

We can use hexahadrons to partition any polyhedral domain Ω in $\mathbb{R}^3$.

Proof.

First for any tetrahedron T, one can decompose it into 4 hexahadrons.



As one can use tetrahedrons to partition any polyhedral domain $\Omega \in \mathbb{R}^3$, it follows that we can use hexadedrons to partition Ω. $\square$

Hexahedral Spline Space

Let $\mathbf{\Delta}$ be a collection of hexahedrons which partitions a polyhedral domain $\Omega \in \mathbb{R}^3$. Let

$$\mathcal{S}_d(\mathbf{\Delta}) = \{s \in C(\Omega), s|_H \in \Psi_d(P_n), \forall H \in \mathbf{\Delta}\}, \tag{14}$$

where $\Psi_d(H)$ is the reduced BB function space for $d \geq 1$. With a straightforward extension, we are able to prove the following

Theorem

For any fixed generalized barycentric coordinates, let $\Psi_2(H)$ be the space spanned by the reduced Bernstein-Bézier functions. Then the dimension of $\mathcal{S}_2(\mathbf{\Delta})$ is

$$\textit{dim}(\mathcal{S}_2(\mathbf{\Delta})) = \#(V) + \#(E),$$

where V is the collection of all vertices of $\mathbf{\Delta}$ and E is the collection of all edges of $\mathbf{\Delta}$.

Numerical Solution of PDE

We now present an application of these polygonal splines for numerical solution of Poisson equation with Dirichlet boundary value problem:

$$\begin{cases} -\Delta u = f, & x \in \Omega \subset \mathbb{R}^2 \\ u = g, & x \in \partial\Omega, \end{cases} \tag{15}$$

where Ω is a polygonal domain and Δ is the standard Laplace operator. Let $S_2(\mathbf{\Delta})$ be the collection of polygonal spline functions of degree 2 over $\mathbf{\Delta}$. Let $B(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$ be the standard bilinear form associated with Poisson equation. Our numerical method is to solve the following weak solution:

$$B(u_h, v) = \langle f, v \rangle, \forall v \in S_2(\mathbf{\Delta}) \cap H_0^1(\Omega). \tag{16}$$

Numerical Solutions over Quadrilaterals

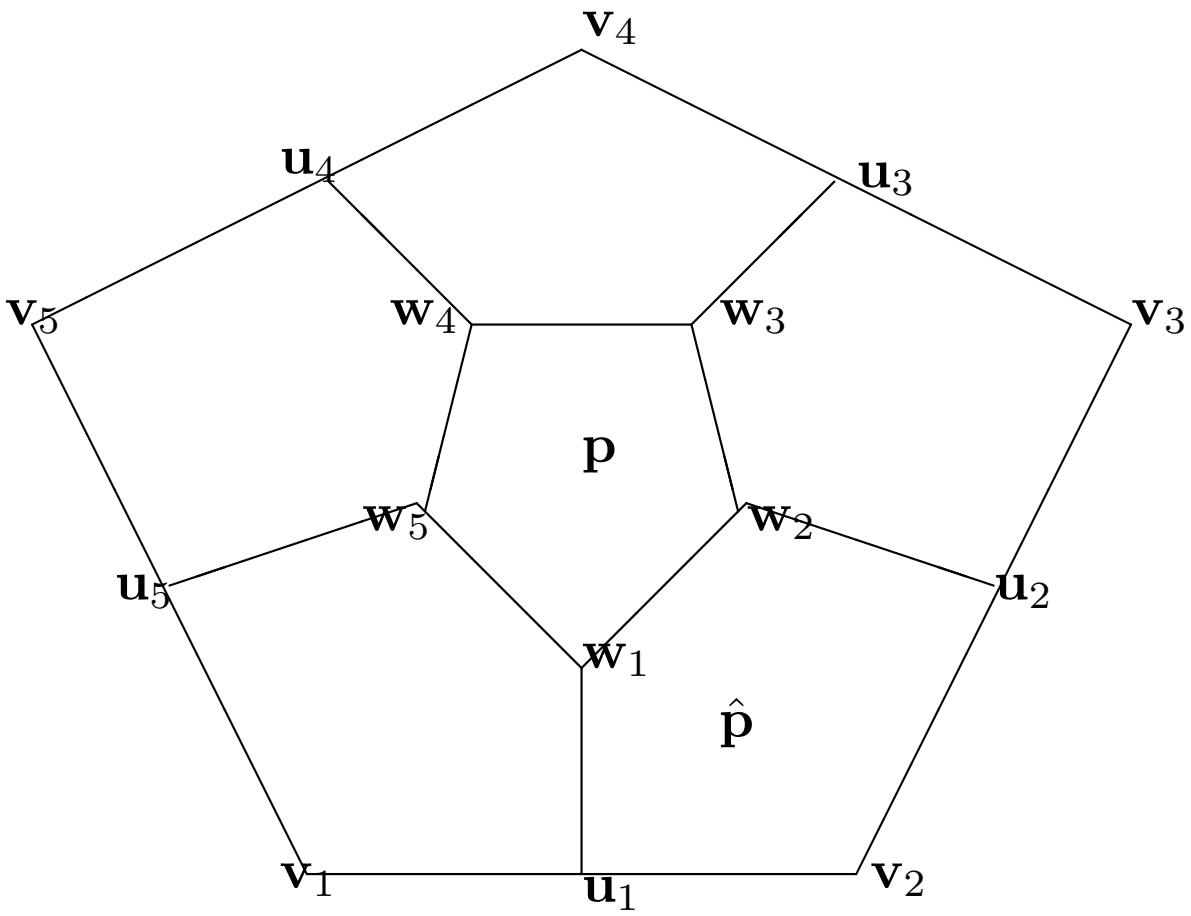
We have done several tests on smooth solutions: $u_1(x,y) = 1/(1+x+y)$, $u_2(x,y) = x^4 + y^4$ and $u_3(x,y) = \sin(\pi(x^2+y^2)) + 1$ to check the performance of our algorithm.

Quads	#DoFs	$\|u_1 - u_{1,h}\|_\infty$	$\|u_2 - u_{2,h}\|_\infty$	$\|u_3 - u_{3,h}\|_\infty$
4	21	0.0084	0.0458	0.3218
16	65	8.5628e-04	0.0065	0.0631
64	225	1.1326e-04	8.6697e-04	0.0084
256	833	1.3895e-05	1.1069e-04	0.0013
1024	3201	1.6907e-06	1.4461e-05	1.5831e-04

Table : Convergence of serendipity quadratic finite elements over uniformly refined quadrangulations

The convergence rate is 3, consistent with the theory and the numerical results from [Rand, Gillette and Babaj, 2014].

Uniform Refinement of Pentagonal Partitions



Partition of Domains

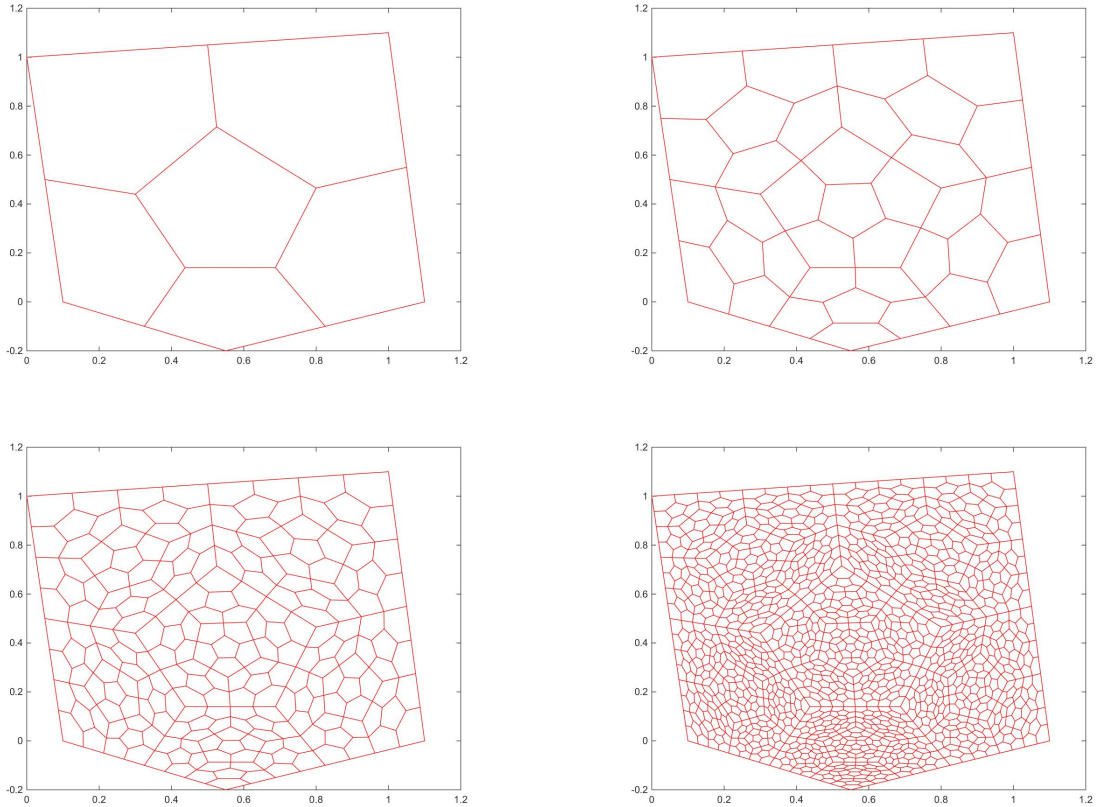


Figure : A Pentagon Partition (top-left) and its refinements

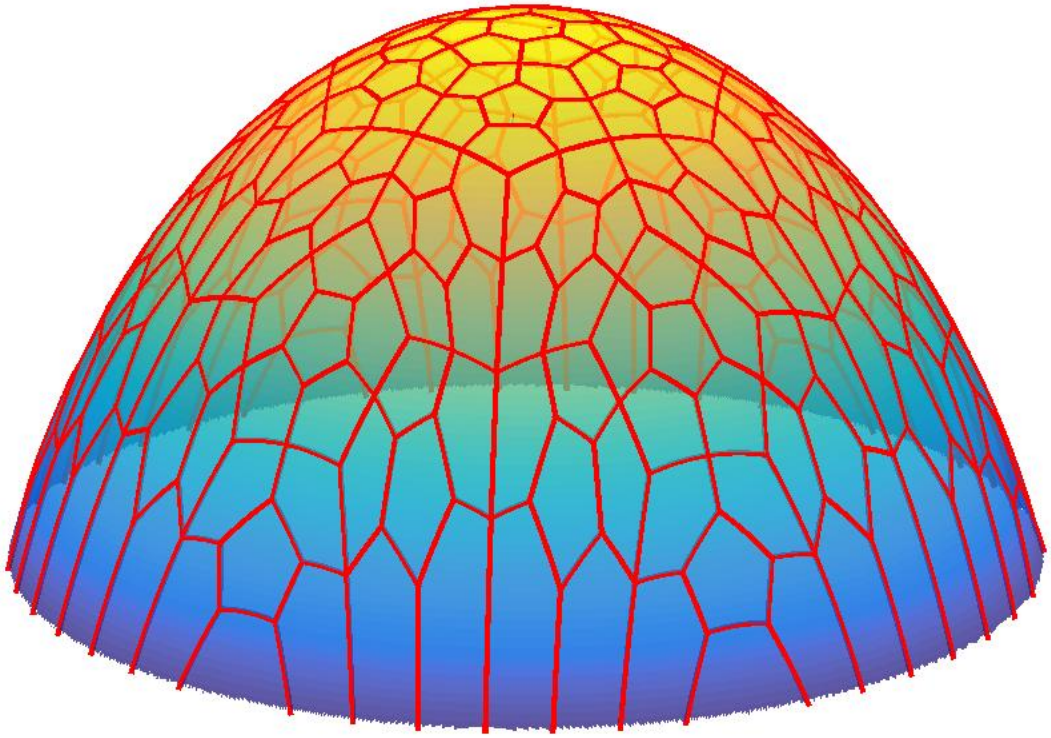
Numerical Solution of PDE over Pentagonal Partitions

pentagons	$\|u_1 - u_{1,h}\|_2$	rates	$\|u_2 - u_{2,h}\|_2$	rates
6	1.0947e-03	0	1.0524e-02	0
36	8.6966e-05	3.65	8.9303e-04	3.55
216	7.4922e-06	3.53	7.5053e-05	3.57
1296	6.4614e-07	3.53	6.3632e-06	3.56
77760	5.9470e-08	3.44	5.5677e-07	3.51

pentagons	$\|u_3 - u_{3,h}\|_2$	rates
6	1.0979e-01	0
36	6.7861e-03	4.01
216	5.6689e-04	3.58
1296	5.0677e-05	3.48
77760	4.4986e-06	3.49

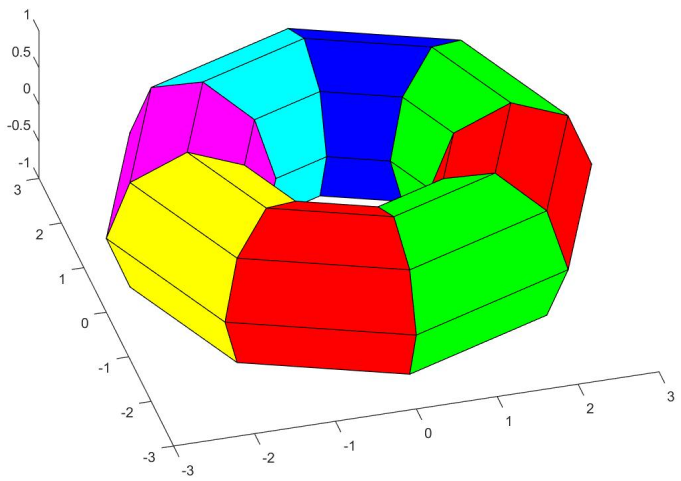
Table : Convergence of C^0 quadratic serendipity finite elements over uniformly refined pentagon partitions in Fig. 2

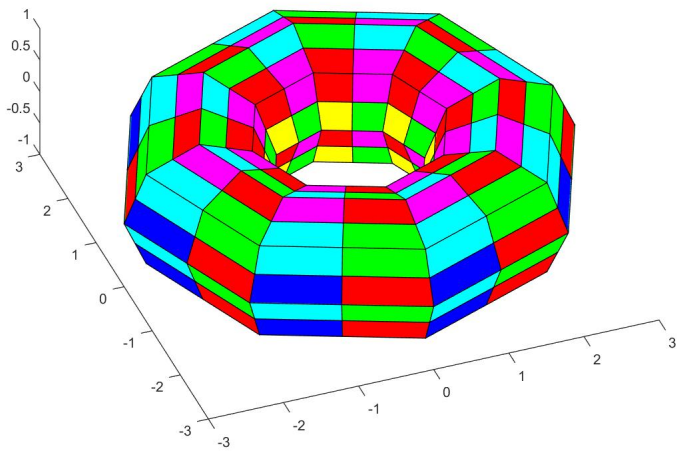
A Numerical Solution over Mixed Polygons



Hexahedral Spline Approximation

Recall the definition of F_i associated with vertex $\mathbf{v}_i$ of a hexahedron H and L_i associated with edge e_i of H. For a collection $\mathbf{\Delta}$ of hexahedraons, we can define a locally suported polyhedral spline function $F_{\mathbf{v}}$ associated with each vertex $\mathbf{v}$ of $\mathbf{\Delta}$ and $L_{\mathbf{e}}$ associated with each edge.





Hexahedral Spline Approximation (II)

We define a quasi-interpolatory operator

$$Q(f) = \sum_{\mathbf{v}\in\mathbf{\Delta}} f(\mathbf{v})F_{\mathbf{v}} + \sum_{\mathbf{e}\in\mathbf{\Delta}} f(\mathbf{u_e})L_{\mathbf{e}}$$

where $\mathbf{u_e}$ is the midpoint of the edge $\mathbf{e}$. Then we have

Theorem

For all quadratic polynomial p in total degree,

$$Q(p) = p.$$

	$Q_f - f$	Rate	D_x	Rate	D_y	Rate	D_z	Rate
$\mathbf{\Delta}$	4.92e-01	0	1.92	0	1.92	0	5.13e-01	0
$\mathbf{\Delta}_1$	6.29e-02	2.96	4.86e-01	1.98	4.86e-01	1.98	1.25e-01	2.03
$\mathbf{\Delta}_2$	8.02e-03	2.97	1.23e-01	1.98	1.23e-01	1.98	3.11e-02	2.0

Table : Convergence C^0 quadratic hexahedral splines to $x^3 + y^3 + z^3$

Conclusion and Future Research

There are a lot of research remaining open:

- dimensions of polygonal BB functions, dimension of polygonal spline space for other GBC's $d \geq 3$ than Wachspress coordinates.
- how to define C^r smooth spline spaces with $r \geq 1$, what are the approximation power of those spline spaces.
- Why is the approximation rate based on pentagonal refinement scheme faster?
- Refinability: let $\mathbf{\Delta}_2$ be a uniform refinement of $\mathbf{\Delta}$. Is $S_d(\mathbf{\Delta}) \subset S_d(\mathbf{\Delta}_2)$?
- We are extending our study to the 3D setting , in particular, numerical solution of PDE.

Thanks for your attention!