NUMERICAL INTEGRATION
of HOMOGENEOUS FUNCTIONS and POLYNOMIALS on
POLYTOPES

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Problem statement

Determine

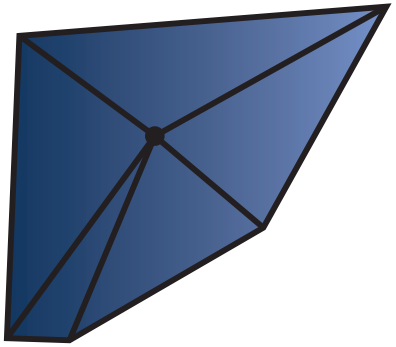
$$\int_{P \subset \mathbb{R}^d} f(\boldsymbol{x})\, d\boldsymbol{x}$$

- $f(\boldsymbol{x})$ is a homogeneous function
- P is a convex or nonconvex polytope

Existing methods

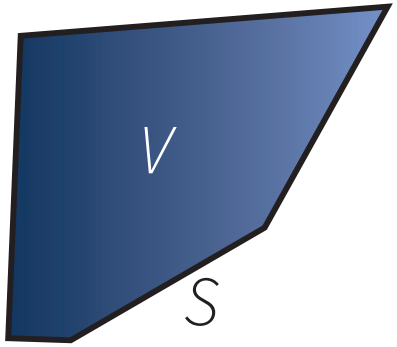
Three methods to integrate functions on polytopes

Triangulation

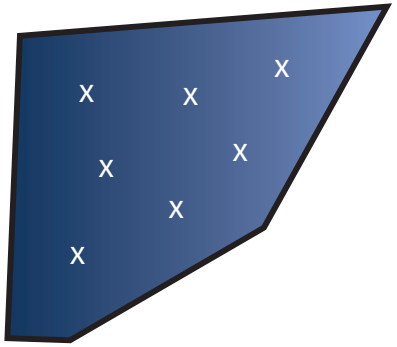


Divergence theorem

$$\int_V \nabla \cdot \boldsymbol{F} \, dV = \int_S \boldsymbol{F} \cdot \boldsymbol{n} \, dS$$



Moment fitting



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Background

Euler's homogeneous func. thm.

A continuously differentiable function $f(\boldsymbol{x})$ is said to be positive homogeneous of degree q if:

$$f(\lambda \boldsymbol{x}) = \lambda^q f(\boldsymbol{x}) \quad \forall \lambda > 0,$$

and then it also satisfies:

$$q f(\boldsymbol{x}) = \langle \nabla f(\boldsymbol{x}), \boldsymbol{x} \rangle \quad \forall \boldsymbol{x} \in \begin{cases} \mathbb{R}^d & \text{if } q > 0 \\ \mathbb{R}^d \backslash \{\boldsymbol{0}\} & \text{if } q < 0 \end{cases}$$

$\langle \cdot, \cdot \rangle$: inner product $\qquad$ d: dimension

Proof

By definition, a homogeneous function of degree q has the property

$$\lambda^q f(\boldsymbol{x}) = f(\lambda \boldsymbol{x})$$

Define $\boldsymbol{x}' := \lambda \boldsymbol{x}$ and calculate $\frac{\partial}{\partial \lambda}$:

$$q\lambda^{q-1} f(\boldsymbol{x}) = \frac{\partial f}{\partial \boldsymbol{x}'} \cdot \frac{\partial \boldsymbol{x}'}{\partial \lambda} = \frac{\partial f}{\partial \boldsymbol{x}'} \cdot \boldsymbol{x}$$

Let $\lambda = 1$:

$$q f(\boldsymbol{x}) = \frac{\partial f}{\partial \boldsymbol{x}} \cdot \boldsymbol{x} = \langle \nabla f(\boldsymbol{x}), \boldsymbol{x} \rangle \qquad \square$$

- Converse is also readily established

Examples of homogeneous fns.

$q = 0$:

$$f(\boldsymbol{x}) = 1$$

$q = 1$:

$$f(\boldsymbol{x}) = x + y$$

$q = 2$:

$$f(\boldsymbol{x}) = 3x^2 + 2xy$$

Examples of homogeneous fns.

$q = 0$:

$$f(\boldsymbol{x}) = 1$$

$q = 1$:

$$f(\boldsymbol{x}) = x + y$$

$q = 2$:

$$f(\boldsymbol{x}) = 3x^2 + 2xy$$

$q = -\frac{1}{2}$:

$$f(\boldsymbol{x}) = \frac{1}{\sqrt{r}},$$

where

$$r = \sqrt{x^2 + y^2}$$

$q = 0$:

$$f(\boldsymbol{x}) = \cos\theta,$$

where

$$\theta = \tan^{-1}\frac{y}{x}$$

$q = \frac{1}{2}$:

$$f(\boldsymbol{x}) = \sqrt{r}\cos\theta$$

Generalized Stokes's theorem

(See: **Taylor, *PDEs: Basic Theory*, 2011**)

$$\int_M d\omega = \int_{\partial M} \omega$$

$$\Downarrow$$

$$\int_M (\nabla \cdot \boldsymbol{X})\, f(\boldsymbol{x})\, d\boldsymbol{x} + \int_M \boldsymbol{X} \cdot \nabla f(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\partial M} (\boldsymbol{X} \cdot \boldsymbol{n})\, f(\boldsymbol{x})\, d\sigma$$

- $\boldsymbol{X}$: vector field
- M: region of integration
- $d\sigma$: Lebesgue measure on ∂M

Extension of Lasserre's method

History

- Method developed by **Lasserre** (**Proc. Am. Math. Soc., 1998, 1999**) for convex regions
- First applied to X-FEM by **Mousavi and S** (**Comp. Mech., 2011**)
- Extended to nonconvex regions by **Chin et al.** (**Comp. Mech., 2015, doi 10.1007/s00466-015-1213-7**) [PDF]
- Method uses properties of homogeneous functions and generalized Stokes's theorem

Main results

Reducing integration to bdry.

Apply Stokes's theorem with $\boldsymbol{X} := \boldsymbol{x}$ and $f(\boldsymbol{x})$ is q-homogeneous:

$$d \int_P f(\boldsymbol{x})\, d\boldsymbol{x} + \int_P \langle \nabla f(\boldsymbol{x}), \boldsymbol{x} \rangle d\boldsymbol{x} = \sum_{i=1}^{m} \int_{\boldsymbol{F}_i} (\boldsymbol{x} \cdot \boldsymbol{n}_i)\, f(\boldsymbol{x})\, d\sigma$$

P: polygon $\qquad$ $\boldsymbol{F}_i$: boundary facets

Apply Euler's homogeneous fn. theorem, $qf(\boldsymbol{x}) = \langle \nabla f(\boldsymbol{x}), \boldsymbol{x} \rangle$:

$$d \int_P f(\boldsymbol{x})\, d\boldsymbol{x} + q \int_P f(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{i=1}^{m} \int_{\boldsymbol{F}_i} (\boldsymbol{x} \cdot \boldsymbol{n}_i)\, f(\boldsymbol{x})\, d\sigma$$

$$\int_P f(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{d+q} \sum_{i=1}^{m} \int_{\boldsymbol{F}_i} (\boldsymbol{x} \cdot \boldsymbol{n}_i)\, f(\boldsymbol{x})\, d\sigma$$

Integral of $f(\boldsymbol{x})$

$$\int_P f(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{d+q} \sum_{i=1}^{m} \int_{\boldsymbol{F}_i} (\boldsymbol{x} \cdot \boldsymbol{n}_i) f(\boldsymbol{x})\, d\sigma$$

- $\boldsymbol{F}_i \subset \boldsymbol{a}_i \cdot \boldsymbol{x} = b_i$: equation of a hyperplane
- $\boldsymbol{n}_i = \frac{\boldsymbol{a}_i}{\|\boldsymbol{a}_i\|}$: unit normal to hyperplane
- $\boldsymbol{x} \cdot \boldsymbol{n}_i = \boldsymbol{x} \cdot \frac{\boldsymbol{a}_i}{\|\boldsymbol{a}_i\|} = \frac{\boldsymbol{x} \cdot \boldsymbol{a}_i}{\|\boldsymbol{a}_i\|} = \frac{b_i}{\|\boldsymbol{a}_i\|}$

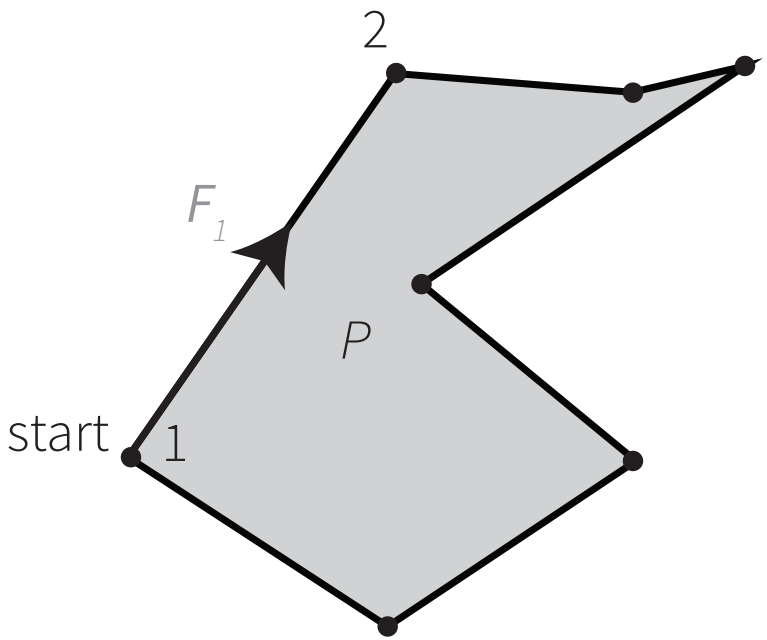
$$\therefore \quad \int_P f(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{d+q} \sum_{i=1}^{m} \int_{\boldsymbol{F}_i} \frac{b_i}{\|\boldsymbol{a}_i\|} f(\boldsymbol{x})\, d\sigma \qquad (*)$$

- Using (*), one can reduce integration to the boundary of the polytope

Sign of a_i and b_i

Question: $x = 1$ and $-x = -1$ produce the same line, yet only one gives the correct answer in (*). Which is correct?

Answer: Given the vertices of the polygon, travel around the polygon in a clockwise direction.



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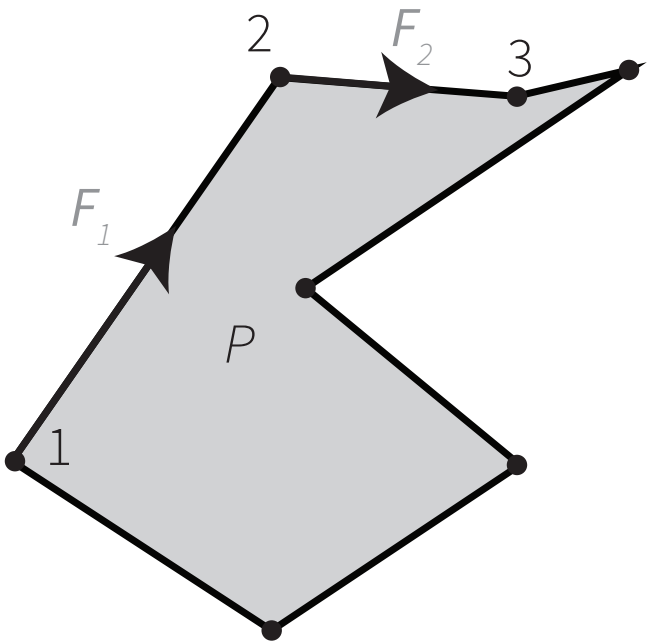

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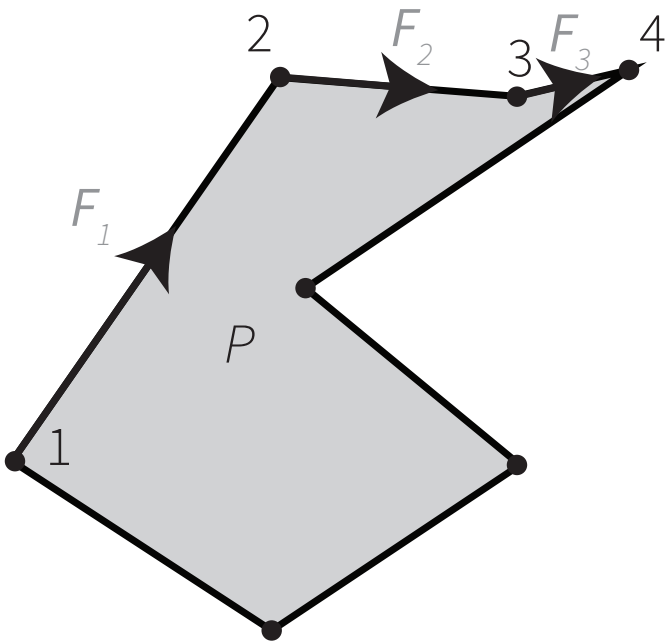

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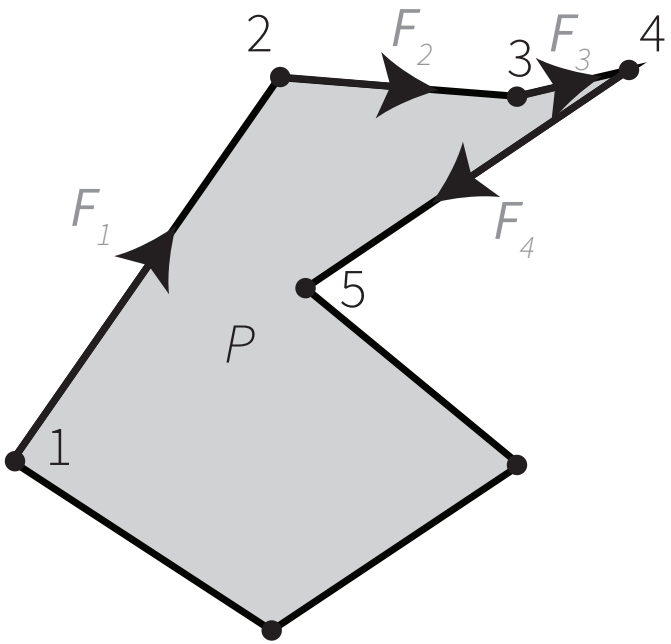

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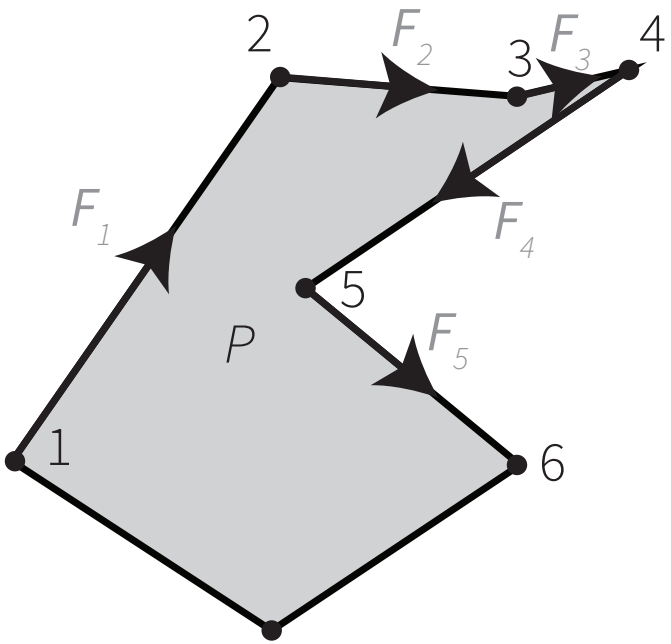

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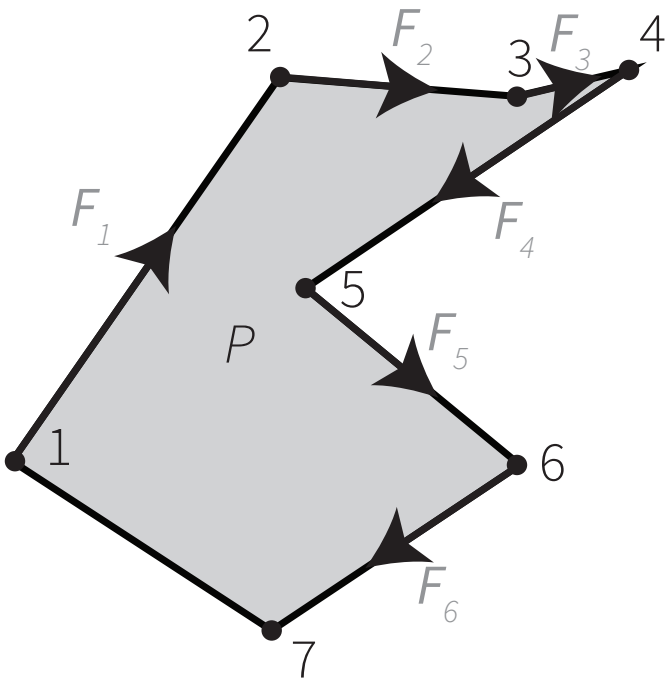

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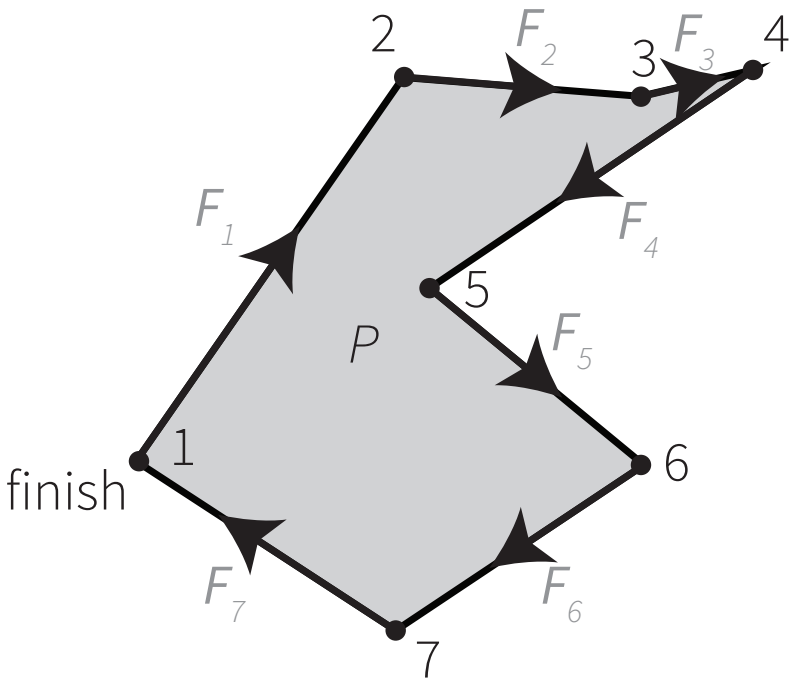

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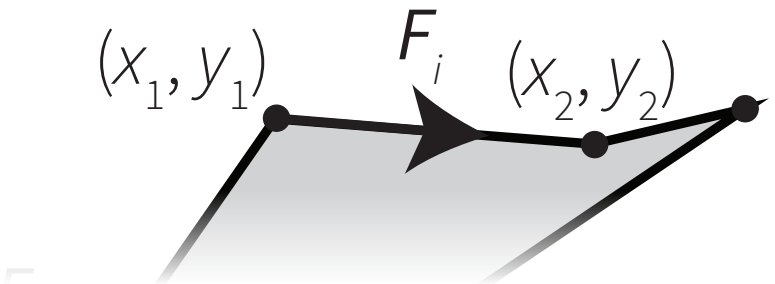


$$\det \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = (y_1 - y_2)x + (x_2 - x_1)y + (x_1 y_2 - y_1 x_2)$$

$$\Downarrow$$

$$\boldsymbol{a}_i = \{y_1 - y_2, x_2 - x_1\}^T \qquad b_i = -(x_1 y_2 - y_1 x_2)$$

Further reducing the integration

Reapplying Stokes's theorem

$$\int_M (\nabla \cdot \boldsymbol{X})\, f(\boldsymbol{x})\, d\boldsymbol{x} + \int_M \boldsymbol{X} \cdot \nabla f(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\partial M} (\boldsymbol{X} \cdot \boldsymbol{n})\, f(\boldsymbol{x})\, d\sigma$$

Select:

- M as $\boldsymbol{F}_i$
- ∂M as $\boldsymbol{v}_{ij}$ (vertices in $\mathbb{R}^2$ – intersection of $\boldsymbol{F}_i$ and $\boldsymbol{F}_j$)
- $\boldsymbol{X} := \boldsymbol{x} - \boldsymbol{x}_0$ ($\boldsymbol{x}_0$: any point on hyperplane containing $\boldsymbol{F}_i$)
- $f(\boldsymbol{x})$ is a homogeneous function of degree q

Integral of $f(x)$ on F_i

Reapplying Stokes's theorem

$$\int_M (\nabla \cdot \boldsymbol{X})\, f(\boldsymbol{x})\, d\boldsymbol{x} + \int_M \boldsymbol{X} \cdot \nabla f(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\partial M} (\boldsymbol{X} \cdot \boldsymbol{n})\, f(\boldsymbol{x})\, d\sigma$$

$$\Downarrow$$

$$\int_{\boldsymbol{F}_i} f(\boldsymbol{x})\, d\sigma = \frac{1}{d+q-1}\left[\sum_{j=1}^{2} d_{ij} f(\boldsymbol{v}_{ij}) + \int_{\boldsymbol{F}_i} \langle \nabla f(\boldsymbol{x}), \boldsymbol{x}_0 \rangle\, d\sigma\right] \quad (**)$$

- $d_{ij} := \langle \boldsymbol{x} - \boldsymbol{x}_0, \boldsymbol{n}_{ij} \rangle$ – algebraic distance from $\boldsymbol{v}_{ij}$ to $\boldsymbol{x}_0$
- $\int_{\boldsymbol{F}_i} \langle \nabla f(\boldsymbol{x}), \boldsymbol{x}_0 \rangle\, d\sigma$ can be applied *recursively*

Results

$$\int_P f(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{d+q} \sum_{i=1}^{m} \int_{\boldsymbol{F}_i} \frac{b_i}{\|\boldsymbol{a}_i\|} f(\boldsymbol{x})\, d\sigma \qquad (*)$$

$$\int_{\boldsymbol{F}_i} f(\boldsymbol{x})\, d\sigma = \frac{1}{d+q-1} \left[\sum_{j=1}^{2} d_{ij} f(\boldsymbol{v}_{ij}) + \int_{\boldsymbol{F}_i} \langle \nabla f(\boldsymbol{x}), \boldsymbol{x}_0 \rangle\, d\sigma \right] \qquad (**)$$

- These formulas can be used to reduce integration to the vertices of the polytope
- Further, a closed-form cubature rule can be developed
- If the partial derivatives of $f(\boldsymbol{x})$ eventually vanish, this cubature rule is *exact*

2D closed-form cubature rule

Combine (*) with (**):

$$\int_P f(\boldsymbol{x})\,d\boldsymbol{x} = \frac{\sum_{i=1}^{m} \frac{b_i}{\|\boldsymbol{a}_i\|} \sum_{j\neq i} d_{ij} I(\boldsymbol{v}_{ij})}{(q+2)(q+1)}$$

where

$$I(\boldsymbol{v}_{ij}) := \sum_{k=0}^{q} \frac{Q_k(\boldsymbol{v}_{ij})}{\binom{q}{k}} \quad \text{and} \quad Q_k(\boldsymbol{v}_{ij}) := \sum_{|\alpha|=q-k} \frac{D^{|\alpha|} f(\boldsymbol{v}_{ij})}{\alpha!} \prod_{\ell=1}^{2} (\boldsymbol{x}_{0\ell})^{\alpha_\ell}$$

- α is an n-tuple of nonnegative integers
- D is the differential operator in multi-index notation

Curved regions

Consider a region V bounded by homogeneous functions $h_i(\boldsymbol{x}) = b_i$ $(i = 1, \ldots, m)$ of degree q_i

Apply Stokes's theorem:

$$\int_V f(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{d+q} \sum_{i=1}^{m} q_i b_i \int_{A_i} \|\nabla h_i\|^{-1} f(\boldsymbol{x})\, d\sigma$$

Polar transformation:

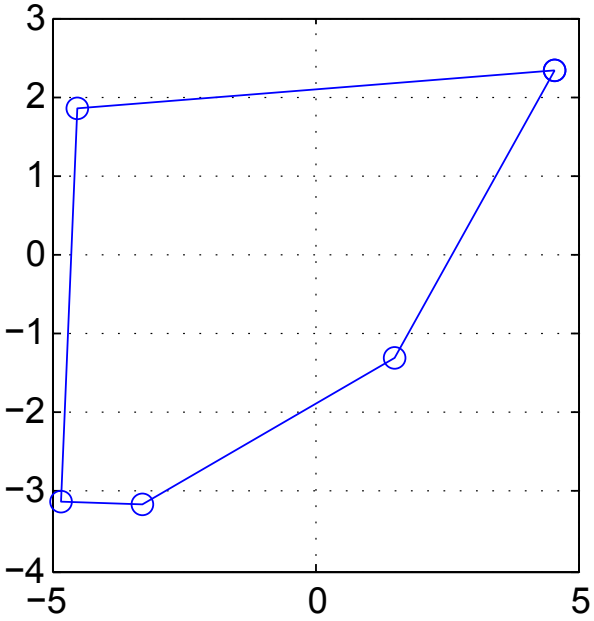
$$\int_V f(\boldsymbol{x})\, d\boldsymbol{x} = \frac{1}{2+q} \sum_{i=1}^{m} \int_{\alpha_i}^{\beta_i} H_i^2(\theta) f(\boldsymbol{x}(\theta))\, d\theta$$

- Region bounded by equations of the form $r = H_i(\theta)$

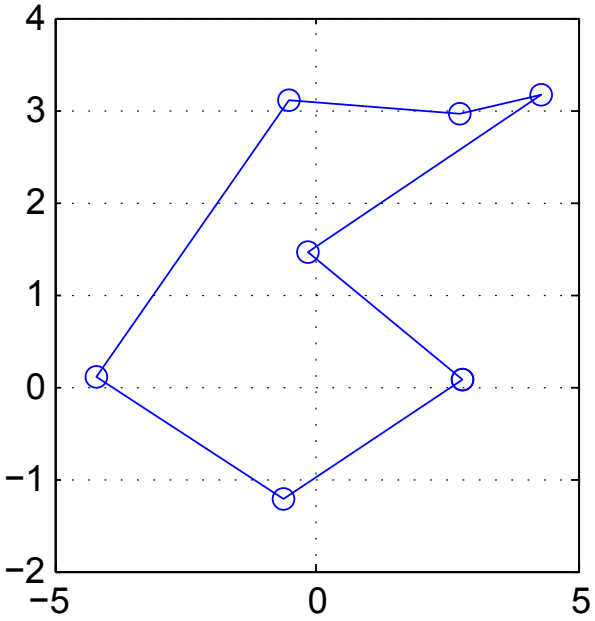
Numerical examples

Integration over polygons

Integrate $f(\boldsymbol{x}) = x^2 + xy + y^2$ over convex and nonconvex polygons



$\int_P f(\boldsymbol{x})\, d\boldsymbol{x} \approx 323.1821$



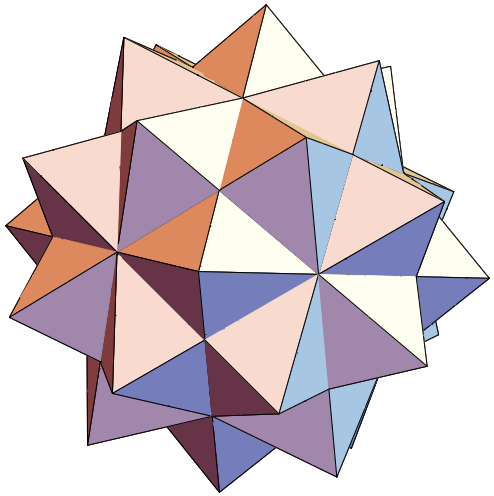
$\int_P f(\boldsymbol{x})\, d\boldsymbol{x} \approx 80.95348$

- Results verified in `LattE` (**De Loera et al., Comput Geom, 2013**)

Integration over polyhedra

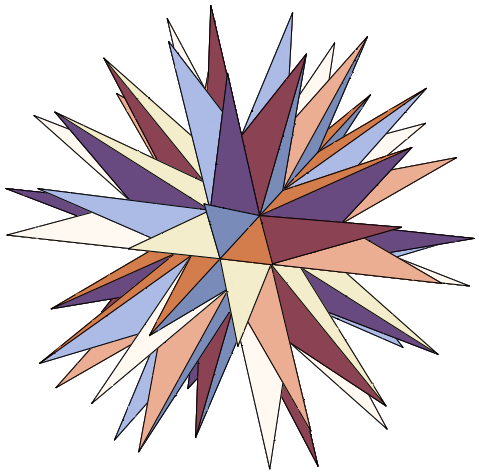
Integrate $f(\boldsymbol{x}) = x^2 + y^2 + z^2$ over nonconvex polyhedra

Octahedron 5-compound



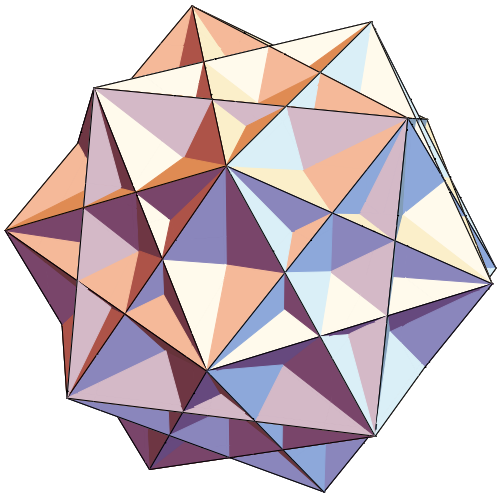
$\int_P f(\boldsymbol{x})\, d\boldsymbol{x} \approx 0.353553$

Echidnahedron



$\int_P f(\boldsymbol{x})\, d\boldsymbol{x} \approx 253.5696$

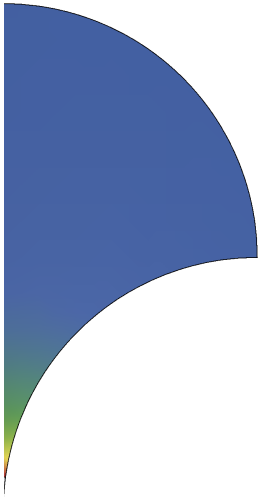
Cube 5-compound



$\int_P f(\boldsymbol{x})\, d\boldsymbol{x} \approx 1.250000$

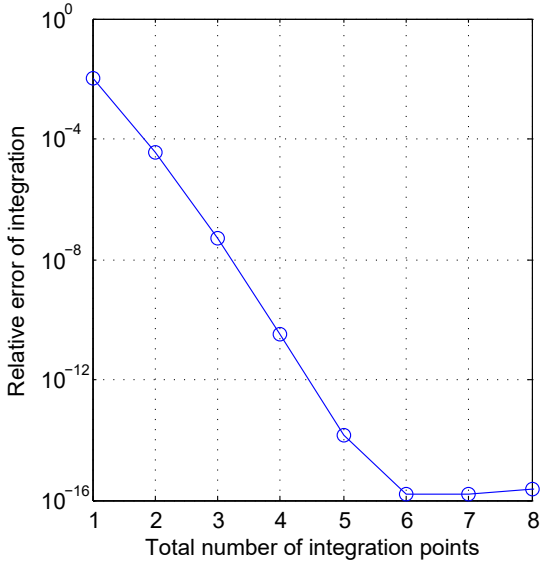
- Shape data from `PolyhedronData[]` in Mathematica

Integration over curved region



$$A := \{r \in [0,1], \theta \in [\pi/4, \pi/2], \\ r \geq \cos\theta, r \leq \sin\theta, \theta \leq \pi/2\}$$

$$\boxed{\int_A \frac{1}{\sqrt{x^2+y^2}} = \sqrt{2} - 1}$$

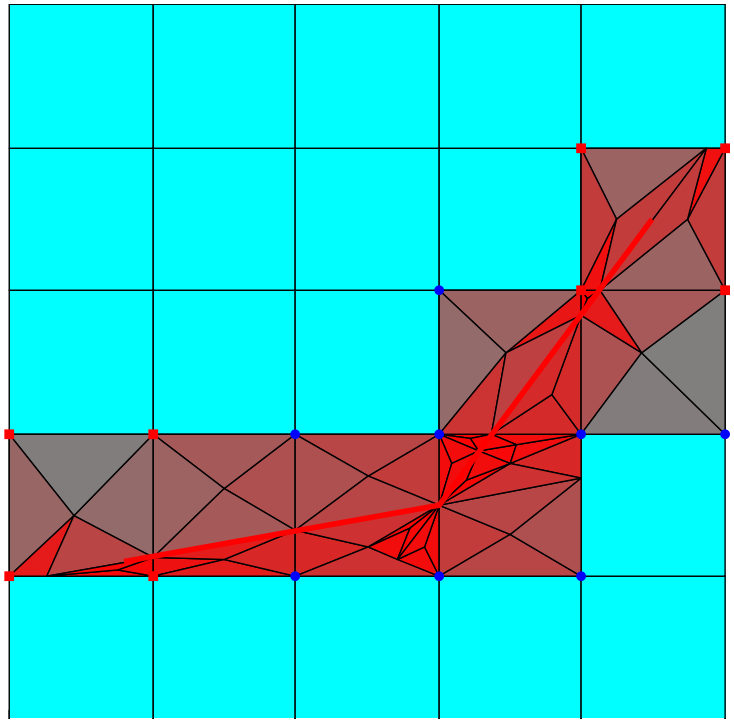


- Weakly singular integrand at the origin
- Using equation derived for a curved region, domain integral is transformed to 1D line integrals
- With 6 quadrature points, integration error is close to machine precision

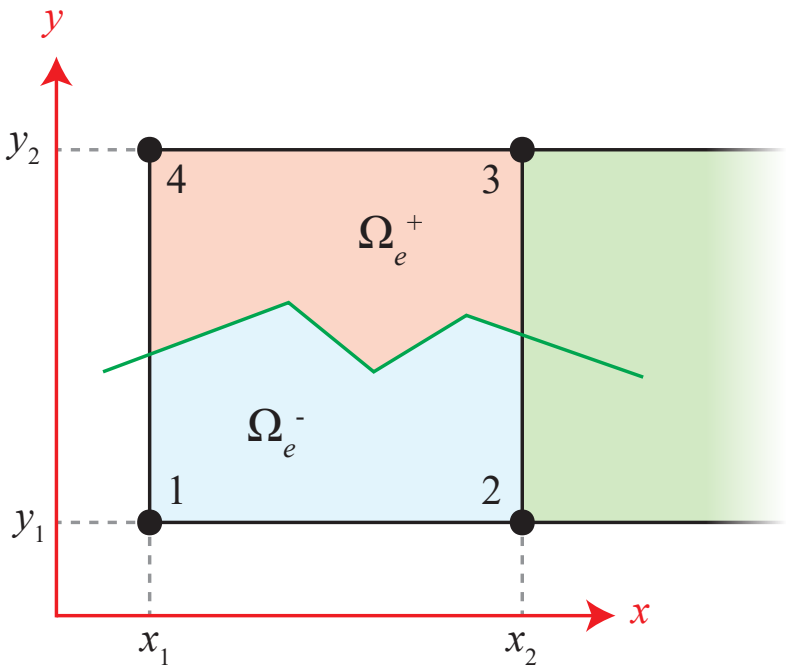
Application: X-FEM (Chin & S, work in progress, 2015)

Motivation

Numerical integration in elements with discontinuous and weakly singular integrands



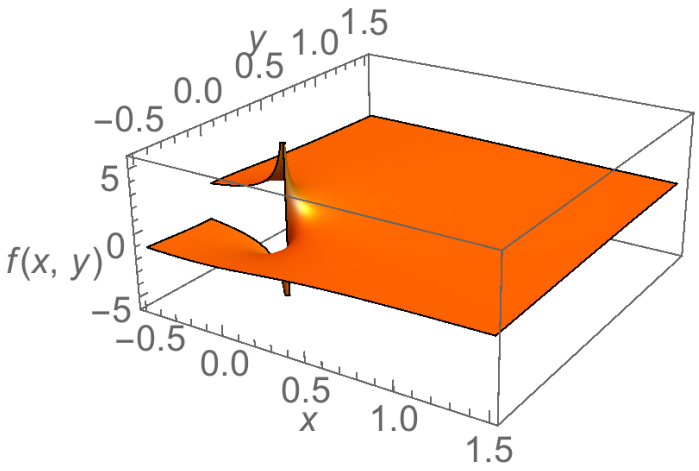
Current approach

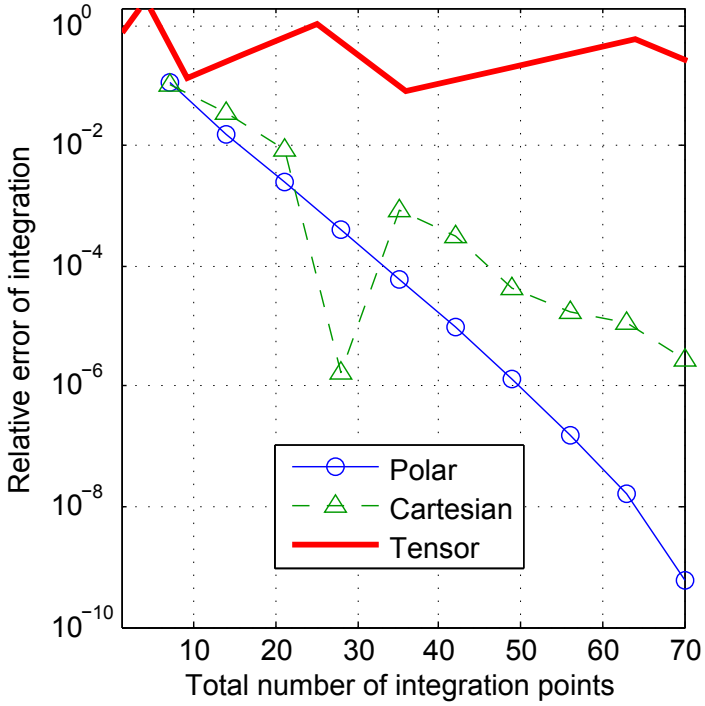


New approach: without partitioning!

Discontinuous + singular integs.

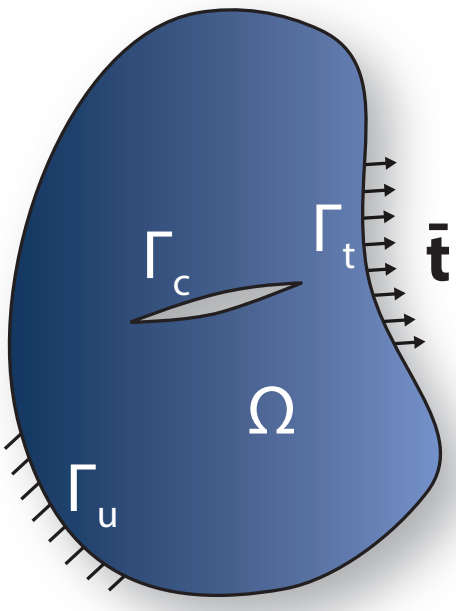
- Use (*) to reduce integration, then apply quadrature
- $f(\boldsymbol{x}) = \sin(\theta/2)/\sqrt{r}$: discontinuous and weakly singular
- Biunit square centered at $(0.5, 0.5)$





Problem formulation

Elastic Fracture



Strong form

$$\nabla \cdot \boldsymbol{\sigma} = 0 \text{ in } \Omega$$
$$\boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on } \Gamma_u$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} \text{ on } \Gamma_t$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_c$$

Weak form

$$a(\boldsymbol{u}, \delta\boldsymbol{u}) = \ell(\delta\boldsymbol{u}) \qquad \forall \delta\boldsymbol{u} \in \mathcal{U}_0,$$

$$a(\boldsymbol{u}, \delta\boldsymbol{u}) := \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\epsilon}\, d\boldsymbol{x}, \quad \ell(\delta\boldsymbol{u}) := \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \delta\boldsymbol{u}\, dS$$

Displacement approximation

Standard FEM

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{i \in \mathbb{I}} N_i(\boldsymbol{x}) \boldsymbol{u}_i$$

Extended FEM [X-FEM] (Moës et al., IJNME, 1999)

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{i \in \mathbb{I}} N_i(\boldsymbol{x}) \boldsymbol{u}_i + \sum_{j \in \mathbb{J} \subseteq \mathbb{I}} N_j(\boldsymbol{x}) \varphi(\boldsymbol{x}) \boldsymbol{a}_j + \sum_{k \in \mathbb{K}_t \subseteq \mathbb{I}} N_k(\boldsymbol{x}) \sum_{t=1}^{2} \sum_{\alpha=1}^{4} F_{\alpha t}(\boldsymbol{x}) \boldsymbol{b}_{k\alpha t}$$

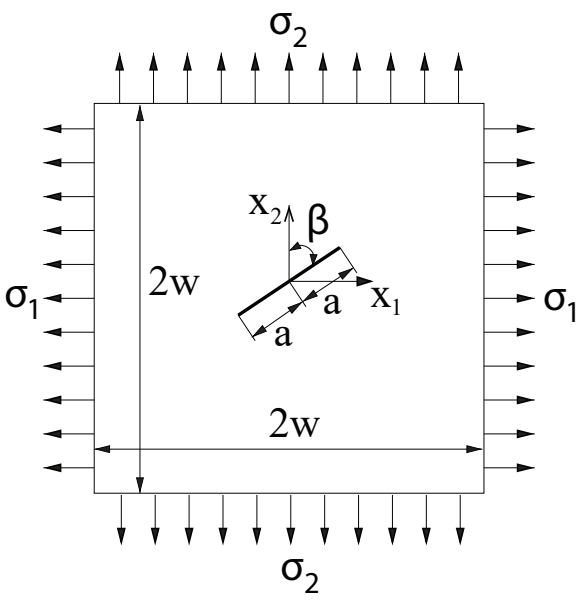
$\boldsymbol{u}_i$, $\boldsymbol{a}_j$, $\boldsymbol{b}_{k\alpha t}$ - degrees of freedom (DOFs)

$\varphi(\boldsymbol{x})$ - discont. enrichment $\left(\text{usually } \varphi(\boldsymbol{x}) := H(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \Omega_e^+ \\ -1 & \boldsymbol{x} \in \Omega_e^- \end{cases} \right)$

$F_{\alpha t}(\boldsymbol{x})$ - crack-tip enrichment

Application: Extraction of SIFs

Mode I SIF for an inclined, embedded crack



β	K_1 (exact)	K_1 (GD)	K_1 (CLS)
$0°$	1.2533	1.2547	1.2549
$15°$	1.3373	1.3379	1.3378
$30°$	1.5666	1.5666	1.5662
$45°$	1.8800	1.8864	1.8861
$60°$	2.1933	2.1936	2.1935
$75°$	2.4227	2.4255	2.4251
$90°$	2.5066	2.5095	2.5088

- Integrand in the interaction integral are homogeneous functions $\Rightarrow$ can use Lassere's approach to compute SIFs
- For the results shown above, Triangulation is used to compute the interaction integral in cracked elements

Conclusions and outlook

Conclusions and outlook

- Presented a method to integrate homogeneous functions over convex and nonconvex polygons and polyhedra using Euler's homogeneous function theorem and generalized Stokes's theorem

Conclusions and outlook

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