Basic Principles of Weak Galerkin Finite Element Methods for PDEs

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Polytopal Element Methods in Mathematics and Engineering

References in Weak Galerkin (WG)

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Talk Outline

1. Basics of Weak Galerkin Finite Element Methods (WG-FEM)
   - weak gradient
   - stabilization (weak continuity)
   - implementation and error analysis
2. An Abstract Framework
3. WG-FEM for Model PDEs
   - mixed formulation
   - hybridized WG
   - linear elasticity
4. Primal-Dual Weak Galerkin – What is it briefly?
   - The Fokker-Planck equation
   - The Cauchy problem for elliptic equations
   - An abstract framework

Related Numerical Methods

1. FEM
2. Stabilized FEMs
3. MFD
4. DG, HDG
5. VEM

Second Order Elliptic Problems

Find $u \in H_0^1(\Omega)$ such that

$$(a\nabla u, \nabla v) = (f, v), \qquad \forall v \in H_0^1(\Omega).$$

Procedures in the standard Galerkin finite element method:

1. Partition Ω into triangles or tetrahedra.
2. Construct a subspace, denoted by $S_h \subset H_0^1(\Omega)$, using piecewise polynomials.
3. Seek for a finite element solution u_h from S_h such that

$$(a\nabla u_h, \nabla v) = (f, v) \qquad \forall v \in S_h.$$

An Out-of-Box Thinking

Replace u_h and v by any distribution, and ∇u_h and ∇v by another distribution, say $\nabla_w v$ as the generalized derivative, and seek for a distribution u_h such that

$$(a\nabla_w u_h, \nabla_w v) = (f, v), \quad \forall v \in S_h.$$

Main Issues:

1. Functions in S_h are to be more general (as distributions or generalized functions) — a good feature
2. The gradient ∇v is computed weakly or as distributions — Questionable and fixable?
3. The numerical approximations are stable and convergent — questionable, how to fix?
4. The schemes are easy to implement and broadly applicable — Ideal, and can be achieved.

Motivation for WG

- The classical gradient ∇u for $u \in C^1(K)$ can be computed as

$$\int_K \nabla u \cdot \phi = -\int_K u \, \nabla \cdot \phi + \int_{\partial K} u(\phi \cdot \mathbf{n})$$

  for all $\phi \in [C^1(K)]^2$.
- The integrals on the right-hand side requires only $u_0 = u$ in the interior of K, plus $u_b = u$ (trace) on the boundary ∂K. We symbolically have

$$\int_K \nabla_w u \cdot \phi = -\int_K u_0 \, \nabla \cdot \phi + \int_{\partial K} u_b \phi \cdot \mathbf{n}$$

Thus, u **can be extended to** $\{u_0, u_b\}$ **with** ∇u **being extended to** $\nabla_w u$**.**

Generalized Weak Derivatives: Foundation of WG

Weak Derivative

For any $u = \{u_0; u_b\}$ with $u_0 \in L^2(K)$ and $u_b \in L^2(\partial K)$, the generalized weak derivative of u in the direction ν is the following linear functional on $H^1(K)$:

$$\langle \partial_\nu u, \phi \rangle = -\int_K u_0 \partial_\nu \phi + \int_{\partial K} (\mathbf{n} \cdot \nu) u_b \phi.$$

for all $\phi \in H^1(K)$.

The **generalized weak derivative** shall be called **weak derivative**.

Weak Functions

Weak Functions

A *weak function* on the region K refers to a generalized function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in L^2(\partial K)$.

- The first component v_0 represents the value of v in the interior of K, and the second component v_b represents v on the boundary of K.
- v_b may or may not be related to the trace of v_0 on ∂K.

The space of weak functions:

$$W(K) = \{v = \{v_0, v_b\} : \ v_0 \in L^2(K), \ v_b \in L^2(\partial K)\}.$$

Weak Gradient

- For any $v \in W(K)$, the **weak gradient** of v is defined as a bounded linear functional $\nabla_w v$ in $H^1(K)$ whose action on each $q \in H^1(K)$ is given by

$$\langle \nabla_w v, q \rangle_K := -\int_K v_0 \nabla \cdot q dK + \int_{\partial K} v_b q \cdot \mathbf{n} ds,$$

  where $\mathbf{n}$ is the outward normal direction on ∂K.
- The weak gradient is identical with the strong gradient for smooth weak functions (e.g., as restriction of smooth functions).

Discrete Weak Gradients

For computational purpose, the weak gradient needs to be approximated, which leads to discrete weak gradients, $\nabla_{w,r}$, given by

$$\int_K \nabla_{w,r} v \cdot q dK = -\int_K v_0 \nabla \cdot q dK + \int_{\partial K} v_b q \cdot \mathbf{n} ds,$$

for all $q \in V(K, r)$. Here

- $V(K, r) \subseteqq [P_r(K)]^2$ is a subspace.
- $P_r(K)$ is the set of polynomials on K with degree $r \geq 0$.
- $V(K, r)$ does not enter into the degrees of freedom in discretization.

Weak Finite Element Spaces

- $\mathcal{T}_h$: **polygonal/polytopal partition** of the domain Ω, shape regular
- construct local discrete elements

$$W_k(T) := \{v = \{v_0, v_b\} : \ v_0 \in P_k(T), v_b \in P_{k-1}(\partial T)\}.$$

- patch local elements together to get a global space

$$S_h := \{v = \{v_0, v_b\} : \ \{v_0, v_b\}|_T \in W_k(T), \forall T \in \mathcal{T}_h\}.$$

Weak FE Space

Weak finite element space with homogeneous boundary value:

$$S_h^0 := \{v = \{v_0, v_b\} \in S_h, v_b|_{\partial T \cap \partial \Omega} = 0, \ \forall T \in \mathcal{T}_h\}.$$

Weak Finite Element Functions

Element Shape Functions for $P_1(K)/P_0(\partial K)$:

$$\phi_i = \{\lambda_i, 0\}, \quad i = 1, 2, 3,$$

$$\phi_{3+j} = \{0, \tau_j\}, \quad j = 1, 2, \cdots, N,$$

where N is the number of sides.

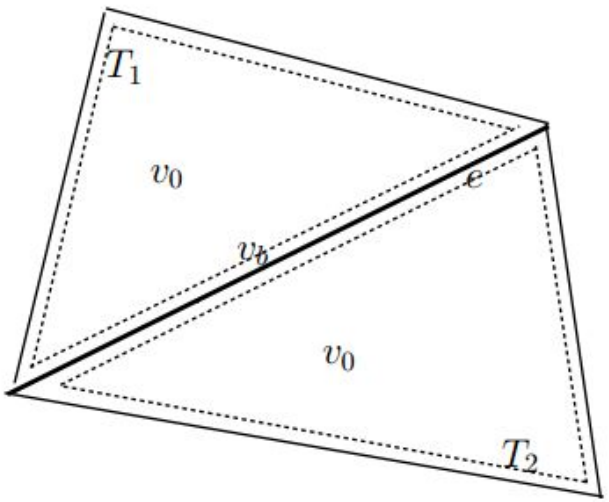
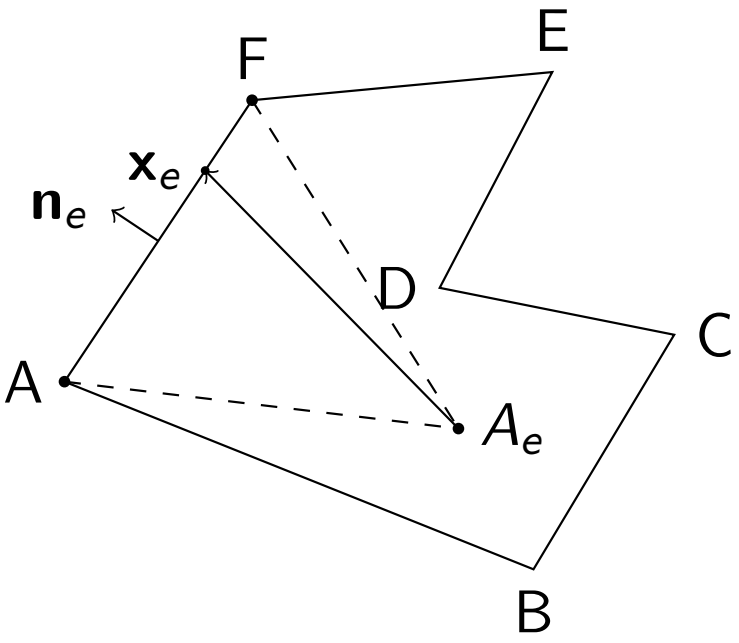


Figure: WG element

Shape Regularity for Polytopal Elements



Why Shape Regularity?

The shape regularity is needed for (1) trace inequality, (2) inverse inequality, and (3) domain inverse inequality.

Weak Galerkin Finite Element Formulation

WG-FEM

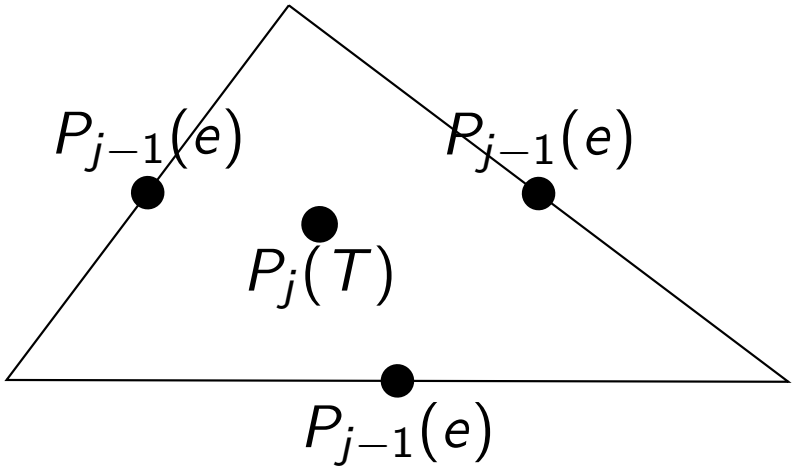
Find $u_h = \{u_0; u_b\} \in S_h^0$ such that

$$(a\nabla_w u_h,\ \nabla_w v) + s(u_h, v) = (f, v_0), \qquad \forall v = \{v_0; v_b\} \in S_h^0,$$

where

1. $\nabla_w v \in P_{k-1}(T)$ is the discrete weak gradient computed locally on each element,
2. $s(\cdot,\cdot)$ is a stabilizer enforcing a weak continuity,
3. the stabilizer $s(\cdot,\cdot)$ measures the discontinuity of the finite element solution.

Depiction of the General WG Element



The polynomial spaces $P_{j-1}(T)$ or $P_j(T)$ can be used for the computation of the weak gradient ∇_w.

Stabilizer

- Commonly used stabilizer:

$$s(w,v) = \rho \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b w_0 - w_b, Q_b v_0 - v_b \rangle_{\partial T},$$

where Q_b is the L^2 projection onto $P_{k-1}(e), e \subset \partial T$.

- Discrete and computation-friendly stabilizer:

$$s(w,v) = \rho \sum_{T \in \mathcal{T}_h} \sum_{x_j} (Q_b w_0 - w_b)(x_j)\ (Q_b v_0 - v_b)(x_j),$$

where $\{x_j\}$ is a set of (nodal) points on ∂T.

WG from the Minimization Perspective

- The original problem can be characterized as

$$u = \arg\min_{v \in H_0^1(\Omega)} \left(\frac{1}{2}(a\nabla v, \nabla v) - (f, v) \right)$$

- Weak Galerkin finite element scheme

$$u_h = \arg\min_{v \in S_h^0} \left(\frac{1}{2}(a\nabla_w v, \nabla_w v) - (f, v_0) + \frac{1}{2}s(v, v) \right).$$

L^2 Projections and Weak Gradients

The following is a commutative diagram:

$$\begin{array}{ccccc} H^1(T) & \xrightarrow{\nabla} & [L^2(T)]^d & & \\ Q_h \big\downarrow & & \mathbb{Q}_h \big\downarrow & & \\ S_h(T) & \xrightarrow{\nabla_w} & V(r,T) & \longrightarrow & 0 \end{array}$$

or equivalently

$$\nabla_w(Q_h u) = \mathbb{Q}_h(\nabla u), \qquad \forall u \in H^1(T).$$

Implication:

- The discrete weak gradient is a good approximation of the true gradient operator.

Error Estimate

- With the correct regularity assumptions, one has the following optimal order error estimate

$$\|Q_h u - u_0\|_0 + h\|Q_h u - u_h\|_{1,h} \leq Ch^{k+1}\|u\|_{k+1}.$$

- Error estimates in negative norms hold true as well. Thus, the WG-FEM solutions are of superconvergent at certain places.

Computational Implementation

1. The stiffness matrix can be assembled as the sum of element stiffness matrices
2. WG preserves physical quantities of importance: mass conservation, energy conservation, etc
3. Suitable for parallel computation; element degree of freedoms can be eliminated in parallel
4. Suitable for multiscale analysis
5. Ideal for problems with discontinuous solutions

An Abstract Framework

Abstract Problem

Find $u \in V$ such that

$$a(u, v) = f(v), \qquad \forall v \in V.$$

In application to PDE, the space V has certain embedded "continuities", such as H^1, $H(div)$, $H(curl)$, H^2, or weighted-version of them. Assume

- V_h: finite dimensional spaces that approximate V
- $a_h(\cdot, \cdot)$: bilinear forms on $V_h \times V_h$ that approximate $a(\cdot, \cdot)$
- f_h: linear functionals on V_h that approximate f.
- $s_h(\cdot, \cdot)$: stabilizers that provide necessary "smoothness"

In WG for Poisson equation, the first bilinear form refers to

$$a_h(u, v) = (a\nabla_w u, \nabla_w v),$$

and the second one $s_h(\cdot, \cdot)$ is the stabilizer.

Abstract WG

Abstract WG

Find $u_h \in V_h$ such that

$$a_h(u_h, v) + s_h(u_h, v) = f_h(v), \quad \forall\ v \in V_h.$$

Some Assumptions:

- *Regularity:* The solution of the abstract problem lies in a subspace $H \subset V$
- The (discrete) norm $\|\cdot\|_{V_h}$ can be extended to $H + V_h$ so that the topology of H is given by the family of semi-norms $\|\cdot\|_{V_h}, h \in (0, h_0)$
- *Boundedness and Coercivity:* The bilinear form

$$a_{wh}(\cdot,\cdot) := a_h(\cdot,\cdot) + s_h(\cdot,\cdot)$$

is bounded and coercive in V_h.

Almost Consistency

The Abstract WG algorithm is said to be *almost consistent* if there exists a linear projection/interpolation operator $Q_h : V \longrightarrow V_h$ such that for each $u_f \in H$ of the abstract problem, one has

- Interpolation approximation:

$$\lim_{h \to 0} \|u_f - Q_h u_f\|_{V_h} = 0$$

- Residual consistency:

$$\lim_{h \to 0} \sup_{v \neq 0, v \in V_h} \frac{|a_h(Q_h u_f, v) - f_h(v)|}{\|v\|_{V_h}} = 0$$

- Almost smoothness:

$$\lim_{h \to 0} \sup_{v \neq 0, v \in V_h} \frac{|s_h(Q_h u_f, v)|}{\|v\|_{V_h}} = 0.$$

Convergence

Convergence

Assume that the Abstract WG algorithm is *almost consistent*. Furthermore, assume that $a_{wh}(\cdot,\cdot)$ is bounded and coercive in V_h, and the solution to the abstract problem lies in the subspace H. Then, we have

$$\lim_{h\longrightarrow 0} \|u_f - u_h\|_{V_h} = 0.$$

For the second order elliptic problem, the convergence can be interpreted as

$$\lim_{h\longrightarrow 0} \left(\|Q_h(\nabla u_f) - \nabla_w u_h\|_0 + s(u_h, u_h)^{\frac{1}{2}} \right) = 0.$$

WG: a New Paradigm of Discretization

Primal Formulation

Find u_h such that

$$(a\nabla_w u_h, \nabla_w v) + s(u_h, v) = (f, v), \quad \forall\ v.$$

Key in WG: discrete weak gradient + stabilization to ensure a weak continuity of u_h in H^1.

Primal-Mixed Formulation

Find u_h and q_h such that

$$\begin{aligned} (a^{-1}\mathbf{q}_h, \mathbf{v}) + (\nabla_w u_h, \mathbf{v}) &= 0, \quad \forall\ \mathbf{v} \\ s(u_h, w) - (\mathbf{q}_h, \nabla_w w) &= (f, w), \quad \forall\ w. \end{aligned}$$

Key in WG: discrete weak gradient + stabilization to provide a weak continuity for u_h in H^1.

WG: a New Paradigm of Discretization

Dual-Mixed Formulation

Find $\mathbf{q}_h$ and u_h such that

$$\begin{aligned} s(\mathbf{q}_h, \mathbf{v}_h) + (a^{-1}\mathbf{q}_h, \mathbf{v}) - (u, \nabla_w \cdot \mathbf{v}) &= 0, \quad \forall\, \mathbf{v} \\ (\nabla_w \cdot \mathbf{q}_h, w) &= (f, w), \quad \forall\, w. \end{aligned}$$

Key in WG: discrete weak divergence + stabilization in velocity to ensure a weak continuity with respect to $H(div)$.

- The space $P_{j+1}(T)$ is used for the computation of $\nabla_w \cdot \mathbf{v}$.
- Lowest order element: pw constant for flux, pw linear for pressure
- The finite element partition is of general polytopal type.

WG Can Be Hybridized

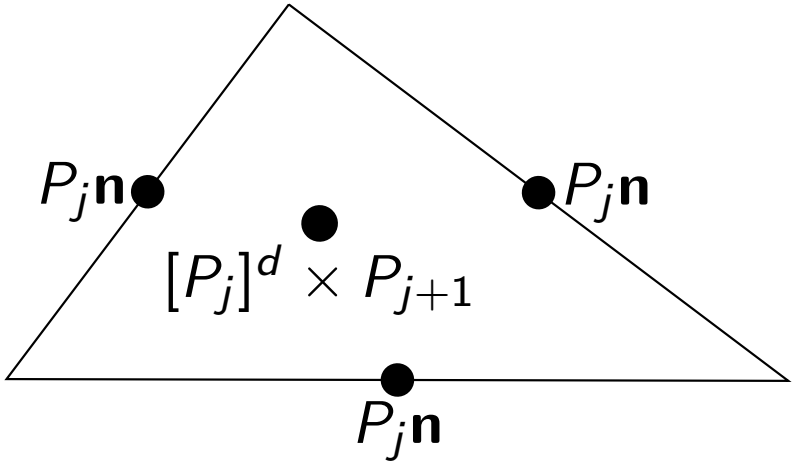
Hybridized Dual-Mixed Formulation

Find $\mathbf{q}_h$, u_h, and λ_h such that

$$s(\mathbf{q}_h, \mathbf{v}_h) + (a^{-1}\mathbf{q}_h, \mathbf{v}) - (u, \nabla_w \cdot \mathbf{v}) + \sum_T \langle \lambda_h, \mathbf{v}_b \cdot \mathbf{n} \rangle_{\partial T} = 0,$$

$$(\nabla_w \cdot \mathbf{q}_h, w) + \sum_T \langle \sigma, \mathbf{q}_b \cdot \mathbf{n} \rangle_{\partial T} = (f, w).$$

Key in HWG: variable reduction in the sense that $\mathbf{q}_h$ and u_h can be eliminated locally on each element.



Linear Elasticity Problems

Model Problem: Find a displacement vector field $\mathbf{u}$ satisfying

$$\begin{aligned} -\nabla \cdot \sigma(\mathbf{u}) &= \mathbf{f}, \qquad \text{in } \Omega, \\ \mathbf{u} &= \widehat{\mathbf{u}}, \qquad \text{on } \Gamma. \end{aligned}$$

Stress-strain relation for linear, homogeneous, and isotropic materials:

$$\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I},$$

(Primal Form) Find $\mathbf{u} \in [H^1(\Omega)]^d$ satisfying $\mathbf{u} = \widehat{\mathbf{u}}$ on Γ and

$$2(\mu\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) + (\lambda\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in [H_0^1(\Omega)]^d.$$

Linear Elasticity Problems-Mixed Formulation

Introducing a pressure variable $p = \lambda \nabla \cdot \mathbf{u}$, the elasticity problem can be reformulated as follows:

(**Mixed Formulation**) Find $\mathbf{u} \in [H^1(\Omega)]^d$ and $p \in L^2(\Omega)$ satisfying $\mathbf{u} = \widehat{\mathbf{u}}$ on Γ, the compatibility condition $\int_\Omega \lambda^{-1} p dx = \int_\Gamma \widehat{\mathbf{u}} \cdot \mathbf{n} ds$,

$$
\begin{aligned}
2(\mu\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) + (\nabla \cdot \mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}), \qquad && \forall \mathbf{v} \in [H_0^1(\Omega)]^d, \\
(\nabla \cdot \mathbf{u}, q) - (\lambda^{-1} p, q) &= 0, && \forall q \in L_0^2(\Omega).
\end{aligned}
$$

Weak Divergence Operator

The space of **weak vector-valued functions** in K

$$V(K) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [L^2(K)]^d, \mathbf{v}_b \in [L^2(\partial K)]^d\}.$$

Weak Divergence

The **weak divergence** of $\mathbf{v} \in V(K)$, $\nabla_w \cdot \mathbf{v}$, is a bounded linear functional on $H^1(K)$, so that its action on any $\phi \in H^1(K)$ is given by

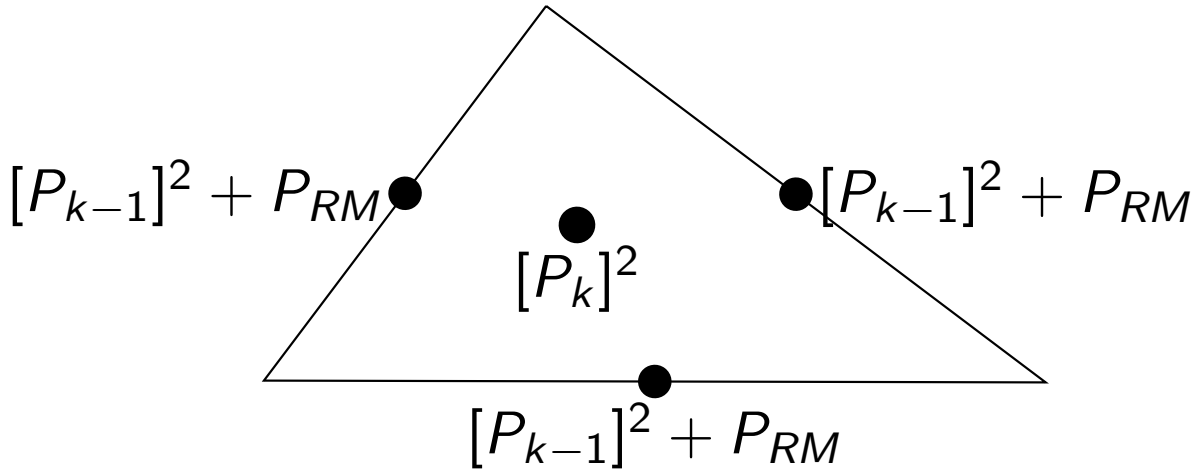
$$\langle \nabla_w \cdot \mathbf{v}, \phi \rangle_K := -(\mathbf{v}_0, \nabla \phi)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, \phi \rangle_{\partial K}.$$

Discrete Weak Divergence

The **discrete weak divergence** of $\mathbf{v} \in V(K)$, denoted by $\nabla_{w,r,K} \cdot \mathbf{v}$, is the unique polynomial, satisfying

$$(\nabla_{w,r,K} \cdot \mathbf{v}, \phi)_K = -(\mathbf{v}_0, \nabla \phi)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, \phi \rangle_{\partial K}, \qquad \forall \phi \in P_r(K).$$

Weak Finite Element Spaces



- $\nabla_w \mathbf{v} \in [P_{k-1}(T)]^{2\times 2}$
- $\nabla_w \cdot \mathbf{v} \in P_{k-1}(T)$.

Weak Galerkin Algorithms for **Primal Formulation**

WG-FEM Primal

Find $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in V_h$ with $\mathbf{u}_b = Q_b\hat{\mathbf{u}}$ on Γ such that for all $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h^0$,

$$\sum_{T\in\mathcal{T}_h} 2(\mu\varepsilon_w(\mathbf{u}_h), \varepsilon_w(\mathbf{v}))_T + (\lambda\nabla_w\cdot\mathbf{u}_h, \nabla_w\cdot\mathbf{v})_T + s(\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}_0).$$

- Q_b: L^2 projection onto $[P_{k-1}(e)]^d + P_{RM}(e)$
- $\varepsilon_w(\mathbf{u}) = \frac{1}{2}(\nabla_w\mathbf{u} + \nabla_w\mathbf{u}^T)$
- Stablizer: $s(\mathbf{w}, \mathbf{v}) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b\mathbf{w}_0 - \mathbf{w}_b, Q_b\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}$

Weak Galerkin Algorithms for Mixed Formulation

WG-FEM in Mixed Form

Find $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in V_h$ and $p_h \in W_h$ satisfying $\mathbf{u}_b = Q_b\hat{\mathbf{u}}$ on Γ, the compatibility condition $(\lambda^{-1}p_h, 1) = \int_\Gamma \widehat{\mathbf{u}} \cdot \mathbf{n} ds$, and

$$2(\mu\varepsilon_w(\mathbf{u}_h), \varepsilon_w(\mathbf{v}))_h + s(\mathbf{u}_h, \mathbf{v}) + (\nabla_w \cdot \mathbf{v}, p_h)_h = (\mathbf{f}, \mathbf{v}_0), \forall \mathbf{v} \in V_h^0,$$
$$(\nabla_w \cdot \mathbf{u}_h, q)_h - \lambda^{-1}(p_h, q) = 0, \forall q \in W_h^0.$$

- $W_h = \{q : \ q|_T \in P_{k-1}(T), \ T \in \mathcal{T}_h\} \quad W_h^0 = W_h \cap L_0^2(\Omega)$

WG-FEM Primal = WG-FEM Mixed

The two weak Galerkin Algorithms are equivalent in the sense that the solutions to the two weak Galerkin Algorithms are identical to each other.

Error Estimate in a Discrete H^1-Norm

Error Estimates and Convergence in H^1

Let the exact solution be sufficiently smooth such that $(\mathbf{u}; p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$. Let $(\mathbf{u}_h; p_h) \in V_h \times W_h$ be the weak Galerkin finite element solution.

$$|||Q_h\mathbf{u}-\mathbf{u}_h|||+\lambda^{-\frac{1}{2}}\|\mathcal{Q}_h p-p_h\|+|||\mathcal{Q}_h p-p_h|||_0 \leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k),$$

where C is a generic constant independent of $(\mathbf{u}; p)$. Consequently,

$$|||\mathbf{u}-\mathbf{u}_h|||+\lambda^{-\frac{1}{2}}\|p-p_h\|+|||p-p_h|||_0 \leq Ch^k(\|\mathbf{u}\|_{k+1}+\|p\|_k).$$

Error Estimate in L^2-Norm

Error Estimates and Convergence in L^2

Assume that the exact solution is sufficiently smooth such that $(\mathbf{u}; p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$. Let $(\mathbf{u}_h; p_h) \in V_h \times W_h$ be the weak Galerkin finite element solution. Then, under the regularity assumption, there exists a constant C, such that

$$\|Q_0\mathbf{u} - \mathbf{u}_0\| \leq Ch^{k+s}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big).$$

Moreover,

$$\|\mathbf{u} - \mathbf{u}_0\| \leq Ch^{k+s}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big).$$

Numerical Results

- $\Omega = (0,1)^2$
- the exact solution $\mathbf{u} = \begin{pmatrix} \sin(x)\sin(y) \\ 1 \end{pmatrix}$

Table: WG based on $\{P_1(T)/P_{RM}(e)\}$, $\lambda = 1$, $\mu = 0.5$.

$1/h$	$\|\mathbf{u}_0 - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0750	–	0.0424	–	0.3103	–
4	0.0192	1.97	0.0115	1.88	0.1566	0.99
8	0.0049	1.98	0.0031	1.87	0.0787	0.99
16	0.0012	1.99	0.0008	1.93	0.0394	1.00
32	0.0003	2.00	0.0002	1.97	0.0197	1.00

Numerical Results

Table: WG based on $\{P_1(T)/P_1(e)\}$, $\lambda = 1$, $\mu = 0.5$.

$\frac{1}{h}$	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|\|$	order
2	0.0743	–	0.0424	–	0.3082	–
4	0.0190	1.96	0.0113	1.90	0.1555	0.99
8	0.0048	1.98	0.0031	1.88	0.0782	0.99
16	0.0012	1.99	0.0008	1.93	0.0392	1.00
32	0.0003	2.00	0.0002	1.97	0.0196	1.00

Numerical Results: Locking-Free Experiments

- $\Omega = (0,1)^2$
- the exact solution

$$\mathbf{u} = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix} + \lambda^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

Table: WG based on $\{P_1(T)/P_{RM}(e)\}$, $\mu = 0.5$, and $\lambda = 1$.

$\frac{1}{h}$	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0352	–	0.0331	–	0.1544	–
4	0.0097	1.86	0.0120	1.46	0.0834	0.89
8	0.0026	1.91	0.0037	1.68	0.0433	0.94
16	0.0007	1.96	0.0010	1.87	0.0220	0.98
32	0.0002	1.98	0.0003	1.96	0.0110	0.99

Numerical Results: Locking-Free Experiments

Table: WG based on $\{P_1(T)/P_{RM}(e)\}$, $\mu = 0.5$, and $\lambda = 1,000,000$.

$\frac{1}{h}$	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0344	–	0.0290	–	0.1447	–
4	0.0100	1.79	0.0113	1.36	0.0773	0.90
8	0.0028	1.82	0.0038	1.59	0.0403	0.94
16	0.0008	1.90	0.0011	1.81	0.0205	0.97
32	0.0002	1.96	0.0003	1.93	0.0103	0.99

Numerical Results: Locking-Free Experiments

Table: WG based on $\{P_1(T)/P_1(e)\}$, $\mu = 0.5$, and $\lambda = 1$.

$\frac{1}{h}$	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0341	–	0.0313	–	0.1518	–
4	0.0093	1.87	0.0115	1.45	0.0816	0.90
8	0.0025	1.91	0.0036	1.67	0.0424	0.95
16	0.0006	1.96	0.0010	1.86	0.0215	0.98
32	0.0002	1.98	0.0003	1.95	0.0108	0.99

Numerical Results: Locking-Free Experiments

Table: WG based on $\{P_1(T)/P_1(e)\}$, $\mu = 0.5$, and $\lambda = 1,000,000$.

$\frac{1}{h}$	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0340	–	0.0280	–	0.1439	–
4	0.0098	1.79	0.0111	1.34	0.0768	0.91
8	0.0028	1.82	0.0037	1.58	0.0400	0.94
16	0.0007	1.90	0.0011	1.81	0.0203	0.97
32	0.0002	1.96	0.0003	1.93	0.0102	0.99

The Fokker-Planck Equation

- describe the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, as in Brownian motion.
- assume the stochastic differential equation:

$$d\mathbf{X}_t = \mu(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{W}_t$$

- the probability density $f(x, t)$ for the random vector $\mathbf{X}_t$ satisfies the Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \nabla \cdot (\mu f) - \frac{1}{2}\sum_{i,j=1}^{N} \partial_{ij}^2 [D_{ij} f] = 0,$$

where $\mu = (\mu_1, \cdots, \mu_N)$ is the drift vector and

$$D_{ij}(x, t) = \sum_{k=1}^{M} \sigma_{ik}(x, t)\sigma_{jk}(x, t)$$

is the diffusion tensor.

Model Problem

Find $u = u(x)$ satisfying

$$\sum_{i,j=1}^{d} \partial_{ij}^2(a_{ij}u) = g, \quad \text{in } \Omega,$$

$$u = 0, \quad \text{on } \partial\Omega.$$

- assume that $a(x)$ is non-smooth,
- weak formulation is given by seeking u such that

$$\sum_{i,j=1}^{d} (u, a_{ij}\partial_{ij}^2 w) = (g, w), \qquad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).$$

A Cauchy Problem for Elliptic Equations

The model problem seeks u such that

$$\begin{aligned} -\Delta u &= f, && \text{in } \Omega, \\ u &= 0, && \text{on } \Gamma \subset \partial\Omega, \\ \frac{\partial u}{\partial n} &= \psi, && \text{on } \Gamma \subset \partial\Omega. \end{aligned}$$

This is usually an ill-posed problem which does not have a solution or has many solutions. Let $\Gamma^c = \partial\Omega/\Gamma$. A variational form for this problem seeks $u \in H^1_{0,\Gamma}(\Omega)$ such that

$$(\nabla u, \nabla w) = (f, w) + \langle \psi, w \rangle_\Gamma,$$

for all $w \in H^1_{0,\Gamma^c}(\Omega)$.

An Abstract Problem

- Let V and W be two Hilbert spaces
- $b(\cdot,\cdot)$ is a bilinear form on $V \times W$
- The *inf-sup* condition of Babuska and Brezzi is satisfied.
- The spaces U and V have certain embedded "continuities", such as L^2, H^1, $H(div)$, $H(curl)$, H^2, or weighted-version of them.

Abstract Problem

Find $u \in V$ such that $b(u, w) = f(w)$ for all $w \in W$. Here f is a bounded linear functional on W.

Goal: Design finite element methods by using weak Galerkin approach.

Specific Examples (Revisited)

- For the Fokker-Planck, we have $V = L^2$ and $W = H^2 \cap H_0^1$ and
$$b(v, w) := \sum_{i,j=1}^{d} (v, a_{ij}\partial_{ij}^2 w).$$
- For the Cauchy problem for Poisson equation, $V \times W = H_{0,\Gamma}^1(\Omega) \times H_{0,\Gamma^c}^1(\Omega)$ and
$$b(v, w) := (\nabla v, \nabla w).$$

Note that the *inf-sup* condition may not be satisfied.

An Abstract Primal-Dual Formulation

- Primal equation in color blue,
- Dual equation in color red,
- They are connected by stabilizers with weak continuity.

WG-FEM

Find $u_h \in V_h$ and $\lambda_h \in W_h$ such that

$$\begin{aligned} s_1(u_h, v) - b_h(v, \lambda_h) &= 0, \qquad \forall v \in V_h \\ s_2(\lambda_h, w) + b_h(u_h, w) &= f_h(w), \quad \forall\, w \in W_h. \end{aligned}$$

- $s_1(\cdot,\cdot)$: stabilizer/smoother in V_h
- $s_2(\cdot,\cdot)$: stabilizer/smoother in W_h

The Primal-Dual WG for Elliptic Cauchy Problem

- the $b_h(\cdot,\cdot)$-form is given by

$$b_h(v,w) := (\nabla_w v, \nabla_w w),$$

- Both stabilizers are given by:

$$s(u,v) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle u_0 - u_b, v_0 - v_b\rangle_{\partial T} + h_T\langle \partial_n u_0 - u_{gn}, \partial_n v_0 - v_{gn}\rangle_{\partial T}.$$

- Error estimates and numerical experiments are on the way.

Numerical Tests

- the right-hand side f is given by
  $2a_{11}+2a_{22}-10a_{12}-10a_{21}-50\sin(30(x-0.5)^2+30(y-0.5)^2)$
- the boundary condition $u = x^2 + 2y^2 - 5xy$ on $\partial\Omega$
- $\Omega = (0,1)^2$

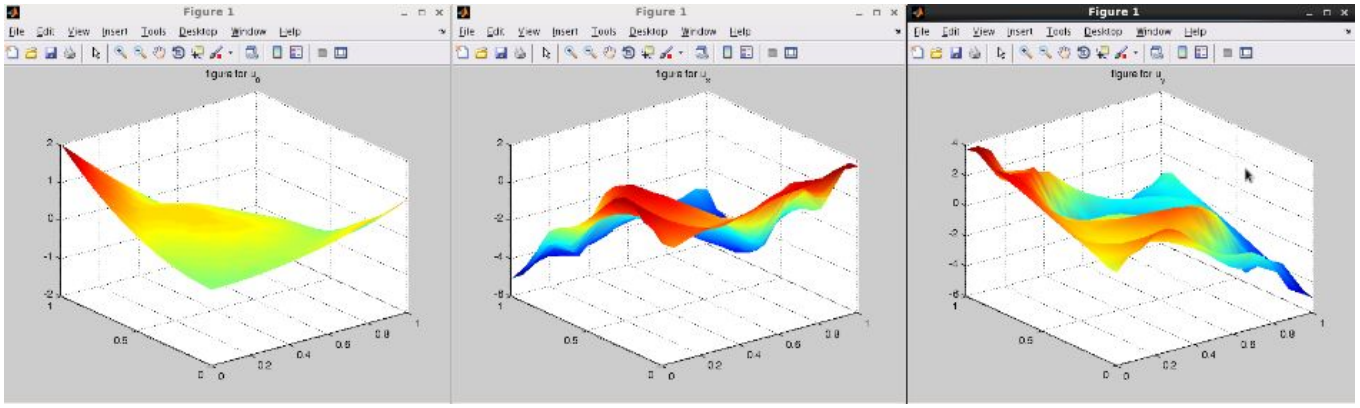


Figure: WG finite element solution with coefficients $a_{11} = 3$, $a_{12} = a_{21} = 1$ and $a_{22} = 2$ in mesh size 1.250e-01 (ρ_h is piecewise linear function).

Numerical Tests

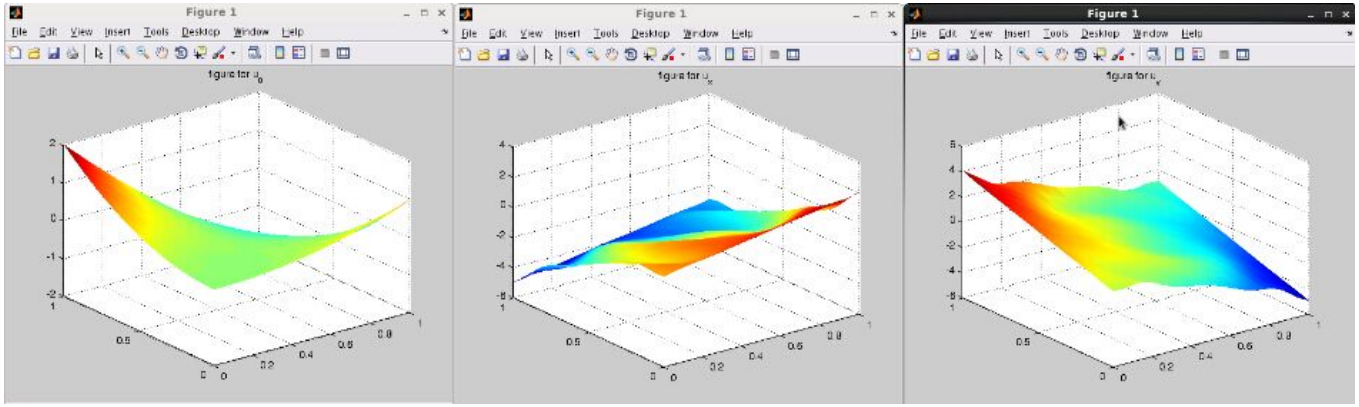


Figure: WG finite element solution with coefficients $a_{11} = 10$, $a_{12} = a_{21} = (0.25x)^2(0.25y)^2$, $a_{22} = 10$ in mesh size 1.250e-01 (ρ_h is piecewise linear function).

Numerical Tests

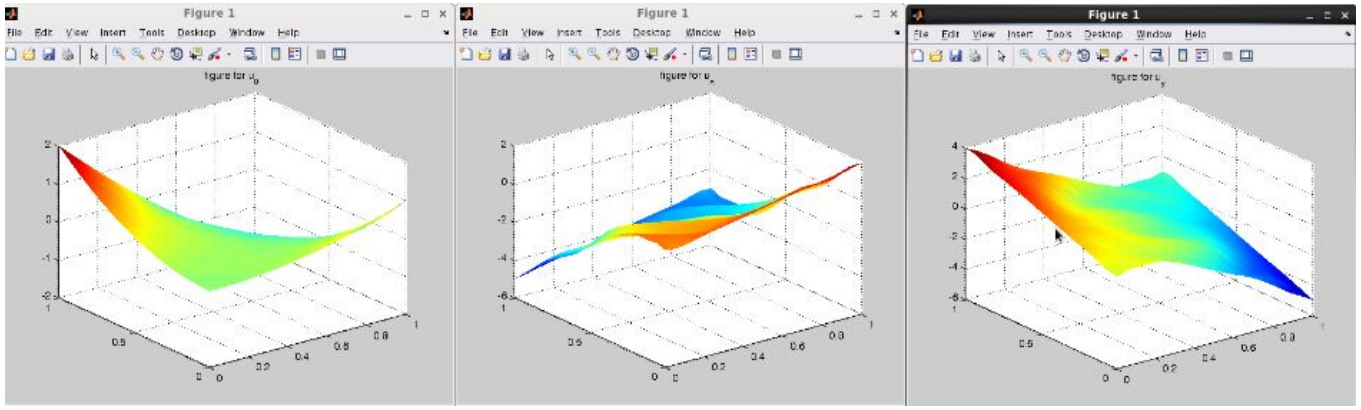


Figure: WG finite element solution with coefficients $a_{11} = 3$, $a_{12} = a_{21} = 1$ and $a_{22} = 2$ in mesh size 1.250e-01 (ρ_h is a piecewise constant).

Numerical Tests

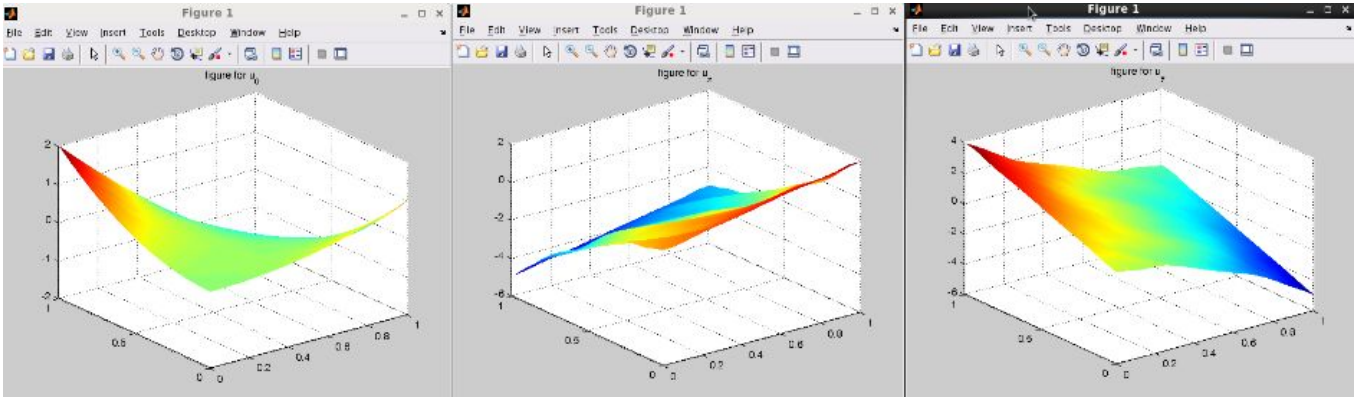


Figure: WG finite element solution with coefficients $a_{11} = 10$, $a_{12} = a_{21} = (0.25x)^2(0.25y)^2$, $a_{22} = 10$ in mesh size 1.250e-01 (ρ_h is a piecewise constant).

Current and Future Research

The current and future research projects include

1. WG on polytopal partitions with curved sides,
2. Fokker-Planck equation,
3. Nonlinear PDEs such as MHD and Cahn-Hillard equations,
4. Variational problems where the trial and test spaces are different, but an inf-sup condition is satisfied,
5. Applications and efficient implementation issues.

Thanks for your attention!