A review of Hybrid High-Order methods: formulations, computational aspects, links with other methods

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Outline

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Lowest-order polytopal discretization methods

Finite Volume methods

- Mixed/Hybrid Finite Volume (M/HFV) [Droniou and Eymard, 06 + Eymard, Gallouët, and Herbin, 10]

Mimetic/Compatible methods

- Mimetic Finite Difference (MFD) [Brezzi, Lipnikov, and Shashkov, 05 + Beirão da Veiga, Lipnikov, and Manzini, 14]

  $\rightsquigarrow$ equivalence with M/HFV [Droniou, Eymard, Gallouët, and Herbin, 10]
- Discrete Geometric Approach (DGA) [Codecasa, Specogna, and Trevisan, 10]
- Compatible Discrete Operator (CDO) [Bonelle and Ern, 14]

Non-conforming/penalized methods

- Cell-Centered Galerkin (CCG) [Di Pietro, 12]
- Generalized Crouzeix–Raviart [Di Pietro and Lemaire, 15]

Unifying frameworks

- Gradient Schemes [Droniou, Eymard, Gallouët, and Herbin, 13]
- CDO

High-order polytopal discretization methods

Finite Element (FE) methods [Wachspress, 75 + Tabarraei and Sukumar, 04 + Gillette, Rand, and Bajaj]

Virtual Element (VE) methods

- Conf. VE [Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, and Russo, 13]
- Non-conf. VE [Ayuso de Dios, Lipnikov, and Manzini]
- Unified framework [Cangiani, Manzini, and Sutton]

Discontinuous Galerkin (DG) methods [Arnold, Brezzi, Cockburn, and Marini, 02 + Di Pietro and Ern, 12 + Bassi, Botti, Colombo, Di Pietro, and Tesini, 12 + Antonietti, Giani, and Houston, 13]

Hybridizable DG (HDG) methods [Cockburn, Gopalakrishnan, and Lazarov, 09]

Weak Galerkin (WG) methods [Wang and Ye, 13]

Hybrid High-Order (HHO) methods [Di Pietro, Ern, and Lemaire, 14]

Outline

Model problem

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be an open, connected, bounded polytopal domain.

Problem: Find a potential $u : \Omega \to \mathbb{R}$ such that

$$\boxed{\begin{aligned} -\operatorname{div}(\mathbb{M}\boldsymbol{\nabla} u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}} \tag{1}$$

$\rightsquigarrow$ $f \in L^2(\Omega)$, $\mathbb{M}$ symmetric, piecewise Lipschitz, matrix-valued coeff. s.t. for a.e. $\boldsymbol{x} \in \Omega$, and all $\boldsymbol{\xi} \in \mathbb{R}^d$ s.t. $|\boldsymbol{\xi}| = 1$,

$$0 < \mu_\flat \leqslant \mathbb{M}(\boldsymbol{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \leqslant \mu_\sharp < +\infty$$

Admissible mesh sequences

Definition

The mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is admissible if, for all $h \in \mathcal{H}$, $\mathcal{T}_h$ is a finite collection of polygons/polyhedra T s.t. $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$, and $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{T}_h$ such that $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the usual sense of Ciarlet;
- contact-regular: every simplex $S \subseteq T$ is s.t. $h_S \approx h_T$.

$\rightsquigarrow$ Assumption: $\mathbb{M} \in \left[\mathbb{P}_d^0(\mathcal{T}_h)\right]_{\mathrm{sym}}^{d\times d}$ $\forall h \in \mathcal{H}$, and $\forall T \in \mathcal{T}_h$, $\mathbb{M}_T := \mathbb{M}_{|T}$ is s.t.

$$\mu_{\flat,T} \leqslant \mathbb{M}_T\boldsymbol{\xi}\cdot\boldsymbol{\xi} \leqslant \mu_{\sharp,T} \quad \text{(local anisotropy ratio: } \rho_T := \mu_{\sharp,T}/\mu_{\flat,T}\text{)}$$

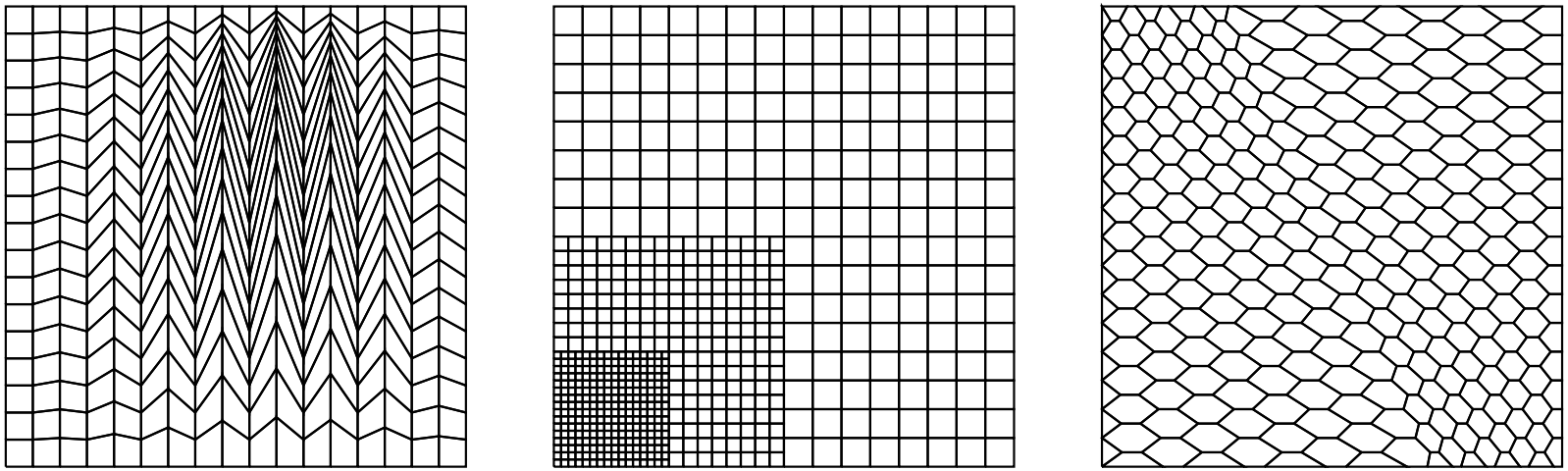

Figure : Admissible meshes in 2D

Outline

HHO in primal form

- Di Pietro, D. A. and Ern, A., A Hybrid High-Order locking-free method for linear elasticity on general meshes, Comput. Meth. Appl. Mech. Engrg., 283:1–21, 2015.

- Di Pietro, D. A., Ern, A., and Lemaire, S., An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators, Comput. Meth. Appl. Math., 14(4):461–472, 2014.

- Di Pietro, D. A. and Ern, A., Hybrid High-Order methods for variable-diffusion problems on general meshes, C. R. Acad. Sci. Paris, Ser. I, 353:31–34, 2015.

Discrete unknowns ($k \geqslant 0$)

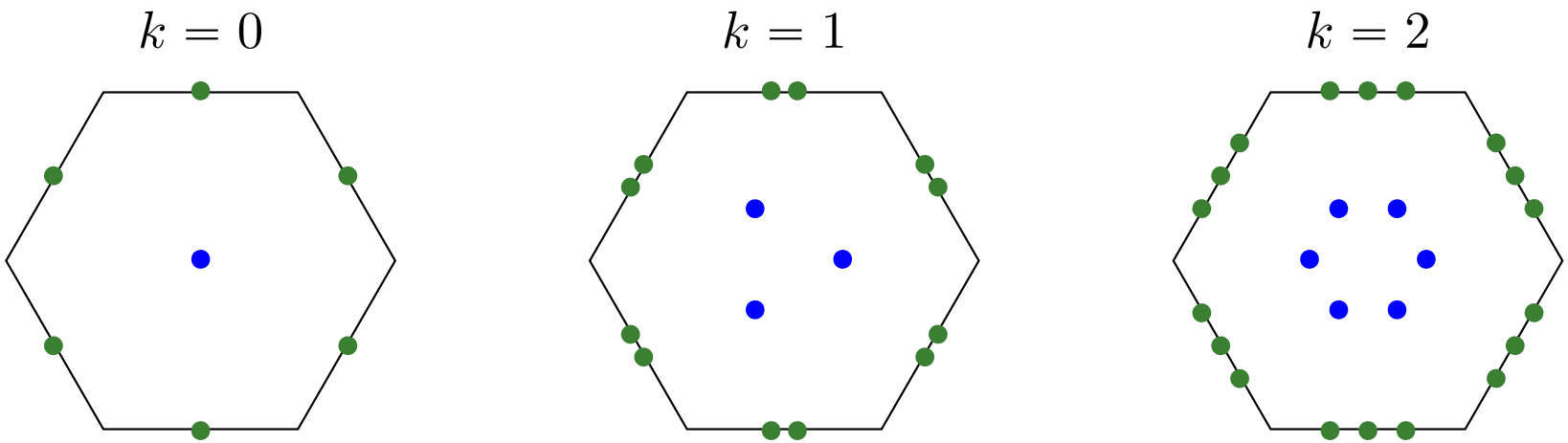

Figure : DoFs associated with potential unknowns, $d = 2$

Local hybrid set of potential unknowns

$$\boxed{\underline{\mathrm{U}}_T^k := \mathbb{P}_d^k(T) \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}}$$

Local reduction operator

$\underline{\mathrm{I}}_T^k : H^1(T) \to \underline{\mathrm{U}}_T^k$ s.t., for all $v \in H^1(T)$, $\quad \underline{\mathrm{I}}_T^k v := \left(\Pi_T^k v, (\Pi_F^k v)_{F \in \mathcal{F}_T} \right)$

Potential reconstruction operator

Local potential reconstruction operator: $\mathrm{p}_T^{k+1} : \underline{\mathrm{U}}_T^k \to \mathbb{P}_d^{k+1}(T)$

For $\underline{\mathrm{v}}_T = (\mathrm{v}_T, \mathrm{v}_{\mathcal{F}_T}) \in \underline{\mathrm{U}}_T^k$, $\mathrm{p}_T^{k+1}\underline{\mathrm{v}}_T \in \mathbb{P}_d^{k+1}(T)$ is s.t. $\int_T \mathrm{p}_T^{k+1}\underline{\mathrm{v}}_T = \int_T \mathrm{v}_T$ and satisfies, for all $\mathrm{w} \in \mathbb{P}_d^{k+1}(T)$,

$$\boxed{(\mathbb{M}_T \boldsymbol{\nabla} \mathrm{p}_T^{k+1}\underline{\mathrm{v}}_T, \boldsymbol{\nabla}\mathrm{w})_T = -(\mathrm{v}_T, \mathrm{div}(\mathbb{M}_T\boldsymbol{\nabla}\mathrm{w}))_T + \sum_{F\in\mathcal{F}_T} (\mathrm{v}_F, \mathbb{M}_T\boldsymbol{\nabla}\mathrm{w}\cdot\boldsymbol{n}_{T,F})_F}$$

$\rightsquigarrow$ diffusivity included in reconstruction operator

Computation

Requires to invert a SPD matrix of size $N_{(k+1),d}$ with $N_{k,l} := \dim(\mathbb{P}_l^k)$

Approximation

For all $v \in H^{k+2}(T)$, the following holds:

$$\|v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v\|_T + h_T\|\boldsymbol{\nabla}(v - \mathrm{p}_T^{k+1}\underline{\mathrm{I}}_T^k v)\|_T \lesssim \rho_T^{1/2} h_T^{k+2}\|v\|_{k+2,T}$$

Stabilization

$$\boxed{a_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := (\mathbb{M}_T \boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{\mathrm{u}}_T, \boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{\mathrm{v}}_T)_T + j_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T)}$$

Local stabilization bilinear form: $j_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}$

For all $\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$,

$$j_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} (\Pi_F^k(\mathrm{q}_T^{k+1} \underline{\mathrm{u}}_T - \mathrm{u}_F), \Pi_F^k(\mathrm{q}_T^{k+1} \underline{\mathrm{v}}_T - \mathrm{v}_F))_F,$$

where $\mu_{T,F} := \mathbb{M}_T \boldsymbol{n}_F \cdot \boldsymbol{n}_F$, and $\mathrm{q}_T^{k+1} \underline{\mathrm{w}}_T := \mathrm{w}_T + (\mathrm{p}_T^{k+1} \underline{\mathrm{w}}_T - \Pi_T^k \mathrm{p}_T^{k+1} \underline{\mathrm{w}}_T)$

$\rightsquigarrow$ the use of Π_F^k is reminiscent of Lehrenfeld-Schöberl stabilization for HDG [Lehrenfeld, 10]

$\rightsquigarrow$ the operator q_T^{k+1} is new and opens the door to lower-order cell unknowns

Approximation

For all $v \in H^{k+2}(T)$, the following bound holds:

$$j_T(\underline{\mathrm{I}}_T^k v, \underline{\mathrm{I}}_T^k v)^{1/2} \lesssim \mu_{\sharp,T}^{1/2} \rho_T^{1/2} h_T^{k+1} \|v\|_{k+2,T}$$

Discrete problem

Global hybrid set of potential unknowns

$$\boxed{\underline{\mathrm{U}}_h^k := \mathbb{P}_d^k(\mathcal{T}_h) \times \mathbb{P}_{d-1}^k(\mathcal{F}_h)}$$

Discrete problem

Find $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_{h,0}^k$ s.t.

$$a_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) = (f, \mathrm{v}_{\mathcal{T}_h}) \quad \text{for all} \quad \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k$$

with $a_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T)$

Stability

$$\rho_T^{-1} \|\mathbb{M}_T^{1/2} \boldsymbol{\nabla} \mathrm{v}_T\|_T^2 + \rho_T^{-1} \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} \|\mathrm{v}_T - \mathrm{v}_F\|_F^2 \lesssim a_T(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T)$$

Error estimates

Theorem (Energy-norm error estimate)

Assume $u \in U_0 \cap H^{k+2}(\mathcal{T}_h)$. *Then,*

$$\|\mathbb{M}^{1/2}(\boldsymbol{\nabla} u - \boldsymbol{\nabla}_h \mathrm{p}_{\mathcal{T}_h}^{k+1} \underline{\mathrm{u}}_h)\| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T} \rho_T^2 h_T^{2(k+1)} \|u\|_{k+2,T}^2 \right\}^{1/2}$$

Theorem (L^2-norm error estimate)

Assume elliptic regularity under the form $\|z(g)\|_{2,\mathcal{T}_h} \lesssim \mu_\flat^{-1} \|g\|$. *Assume* $f \in H^{k+\delta}(\Omega)$, *with* $\delta = 0$ *for* $k \geqslant 1$ *and* $\delta = 1$ *for* $k = 0$. *Then,*

$$\mu_\flat \|\Pi_{\mathcal{T}_h}^k u - \mathrm{u}_{\mathcal{T}_h}\| \lesssim \mu_\sharp^{1/2} \rho\, h \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T} \rho_T^2 h_T^{2(k+1)} \|u\|_{k+2,T}^2 \right\}^{1/2} + h^{k+2} \|f\|_{k+\delta}$$

Local conservativity

1 - Introduce the local bilinear form

$$\hat{a}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) := (\mathbb{M}_T \boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{\mathrm{w}}_T, \boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{\mathrm{v}}_T)_T + \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} (\mathrm{w}_T - \mathrm{w}_F, \mathrm{v}_T - \mathrm{v}_F)_F$$

2 - Define the local isomorphism $\underline{\mathrm{c}}_T^k : \underline{\mathrm{U}}_T^k \to \underline{\mathrm{U}}_T^k$ s.t.

$$\hat{a}_T(\underline{\mathrm{c}}_T^k \underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) = \hat{a}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) + j_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) \qquad \forall\, \underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$$

3 - Define the local gradient recons. operator $\mathbf{G}_T^{k+1} := \boldsymbol{\nabla}(\mathrm{p}_T^{k+1} \circ \underline{\mathrm{c}}_T^k)$

Lemma

For all $T \in \mathcal{T}_h$, the following local equilibrium holds:

$$(\mathbb{M}_T \mathbf{G}_T^{k+1} \underline{\mathrm{u}}_T, \boldsymbol{\nabla} \mathrm{v}_T)_T - \sum_{F \in \mathcal{F}_T} (\Phi_{T,F}(\underline{\mathrm{u}}_T), \mathrm{v}_T)_F = (f, \mathrm{v}_T)_T \qquad \forall\, \mathrm{v}_T \in \mathbb{P}_d^k(T)$$

with conservative numerical flux

$$\Phi_{T,F}(\underline{\mathrm{u}}_T) := \mathbb{M}_T \mathbf{G}_T^{k+1} \underline{\mathrm{u}}_T \cdot \boldsymbol{n}_{T,F} - \frac{\mu_{T,F}}{h_F} \left[(\mathrm{c}_T^k \underline{\mathrm{u}}_T - \mathrm{u}_T) - (\mathrm{c}_F^k \underline{\mathrm{u}}_T - \mathrm{u}_F) \right]$$

Solution strategy

Offline step $\rightsquigarrow$ 2 fully parallelizable and f-independent substeps

- 1 - Compute the potential reconstruction operator $\mathrm{p}_{\mathcal{T}_h}^{k+1}$

  $\rightsquigarrow$ invert $\mathrm{card}(\mathcal{T}_h)$ SPD matrices of size $N_{(k+1),d}$

- 2 - For all $T \in \mathcal{T}_h$, compute the trace-based $\mathrm{t}_T^k : \mathbb{P}_{d-1}^k(\mathcal{F}_T) \to \mathbb{P}_d^k(T)$ and datum-based $\mathrm{d}_T^k : \mathbb{P}_d^k(T) \to \mathbb{P}_d^k(T)$ lifting operators s.t.

$\mathrm{t}_T^k \mathrm{w}_{\mathcal{F}_T} \in \mathbb{P}_d^k(T)$ solves $\quad a_T((\mathrm{t}_T^k \mathrm{w}_{\mathcal{F}_T}, 0), (\mathrm{v}_T, 0)) = -a_T((0, \mathrm{w}_{\mathcal{F}_T}), (\mathrm{v}_T, 0)) \; \forall \mathrm{v}_T \in \mathbb{P}_d^k(T)$

$\mathrm{d}_T^k \Psi_T \in \mathbb{P}_d^k(T)$ solves $\quad a_T((\mathrm{d}_T^k \Psi_T, 0), (\mathrm{v}_T, 0)) = (\Psi_T, \mathrm{v}_T)_T \; \forall \mathrm{v}_T \in \mathbb{P}_d^k(T)$

$\rightsquigarrow$ invert $\mathrm{card}(\mathcal{T}_h)$ SPD matrices of size $N_{k,d}$

Online step

- 1 - Given $f \in L^2(\Omega)$, compute its L^2-orthogonal projection $\Pi_{\mathcal{T}_h}^k f$ onto $\mathbb{P}_d^k(\mathcal{T}_h)$

- 2 - Solve the global problem: Find $\mathrm{u}_{\mathcal{F}_h} \in \mathbb{P}_{d-1,0}^k(\mathcal{F}_h)$ s.t.

$$a_h(\underline{\mathrm{t}}_h^k \mathrm{u}_{\mathcal{F}_h}, \underline{\mathrm{t}}_h^k \mathrm{v}_{\mathcal{F}_h}) = (\Pi_{\mathcal{T}_h}^k f, \mathrm{t}_{\mathcal{T}_h}^k \mathrm{v}_{\mathcal{F}_h}) \qquad \forall \mathrm{v}_{\mathcal{F}_h} \in \mathbb{P}_{d-1,0}^k(\mathcal{F}_h)$$

  where $\underline{\mathrm{t}}_h^k \mathrm{w}_{\mathcal{F}_h} := (\mathrm{t}_{\mathcal{T}_h}^k \mathrm{w}_{\mathcal{F}_h}, \mathrm{w}_{\mathcal{F}_h})$

  $\rightsquigarrow$ solve a linear system of size $\approx \mathrm{card}(\mathcal{F}_h) \times N_{k,(d-1)}$

- 3 - Compute the discrete solution according to $\underline{\mathrm{u}}_h = (\mathrm{t}_{\mathcal{T}_h}^k \mathrm{u}_{\mathcal{F}_h} + \mathrm{d}_{\mathcal{T}_h}^k \Pi_{\mathcal{T}_h}^k f, \mathrm{u}_{\mathcal{F}_h})$

Outline

Links HHO/other polytopal discretization methods

Assume here $\mathbb{M} = \mathrm{Id}_d$.

The HHO(I) family: $\underline{\mathrm{U}}_T^{k,l} := \mathbb{P}_d^l(T) \times \mathbb{P}_{d-1}^k(\mathcal{F}_T)$, $l \in \{k-1, k, k+1\}$

- The choice $l = k$ corresponds to the original HHO method
- The Non-conf. VE method is, up to equivalent stabilization, a member of the HHO family (for $l = k-1$) [Cockburn, Di Pietro, and Ern, 15]
- The final system has the same size for any choice of l

Similarities/differences among discontinuous skeletal methods

- HHO fits into the HDG framework [Cockburn, Di Pietro, and Ern, 15]
- The flux unknowns associated with HDG and WG belong to $\mathbb{P}_d^k(T)^d$, whereas the flux unknowns associated with HHO belong to $\boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T)$: smaller local problems must be solved to eliminate flux unknowns in HHO

Outline

HHO in mixed form

- Di Pietro, D. A. and Ern, A., Arbitrary-order mixed methods for heterogeneous anisotropic diffusion on general meshes, submitted 2013, to appear 2016.

- Aghili, J., Boyaval, S., and Di Pietro, D. A., Hybridization of mixed high-order methods on general meshes and application to the Stokes equations, Comput. Meth. Appl. Math., 15(2):111–134, 2015.

Discrete unknowns ($k \geqslant 0$)

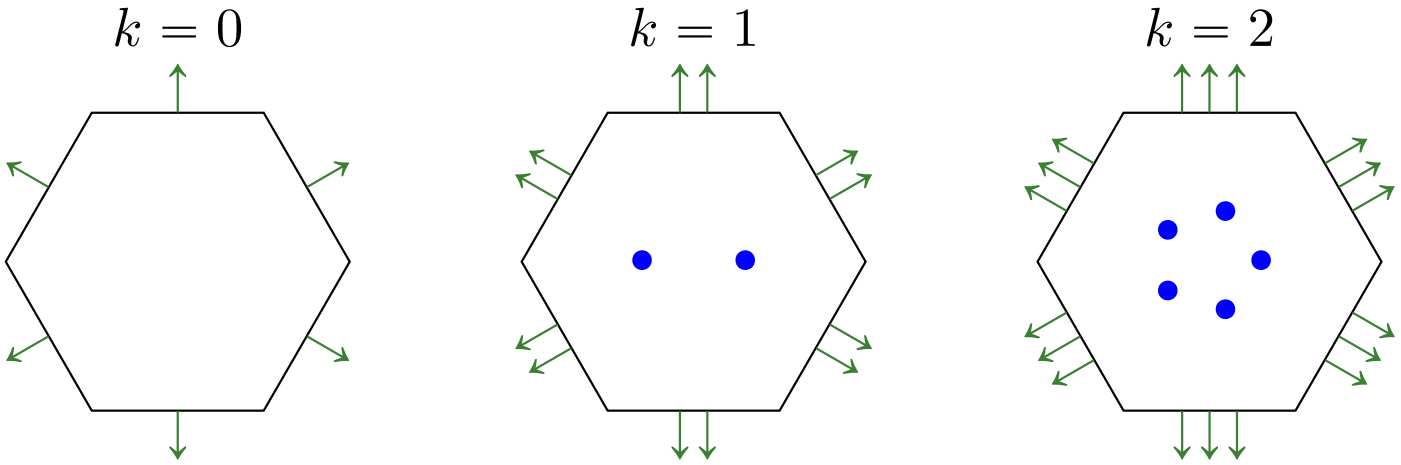

Figure : DoFs associated with flux unknowns, $d = 2$

Local {hybrid set of flux unknowns} + {set of potential unknowns $:= \mathbb{P}_d^k(T)$}

$$\boxed{\underline{\mathbf{S}}_T^k := \mathbb{M}_T \boldsymbol{\nabla} \mathbb{P}_d^k(T) \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}}$$

Local reduction operator: $\boldsymbol{S}^+(T) := \{\boldsymbol{t} \in \boldsymbol{L}^q(T) \mid \operatorname{div} \boldsymbol{t} \in L^2(T)\}$, $q > 2$

$\underline{\mathbf{I}}_T^k : \boldsymbol{S}^+(T) \to \underline{\mathbf{S}}_T^k$ s.t., $\forall \boldsymbol{t} \in \boldsymbol{S}^+(T)$, $\quad \underline{\mathbf{I}}_T^k \boldsymbol{t} := \left(\mathbb{M}_T \boldsymbol{\nabla} \mathrm{y}, (\Pi_F^k(\boldsymbol{t} \cdot \boldsymbol{n}_F))_{F \in \mathcal{F}_T} \right)$,

with $\mathrm{y} \in \mathbb{P}_d^k(T)$ a solution of $(\mathbb{M}_T \boldsymbol{\nabla} \mathrm{y}, \boldsymbol{\nabla} \mathrm{w})_T = (\boldsymbol{t}, \boldsymbol{\nabla} \mathrm{w})_T \quad \forall \mathrm{w} \in \mathbb{P}_d^k(T)$

Divergence reconstruction operator

Local divergence reconstruction operator: $\mathrm{D}_T^k : \underline{\mathbf{S}}_T^k \to \mathbb{P}_d^k(T)$

For all $\underline{\mathbf{t}}_T = (\mathbf{t}_T, \mathrm{t}_{\mathcal{F}_T}) \in \underline{\mathbf{S}}_T^k$, $\mathrm{D}_T^k \underline{\mathbf{t}}_T \in \mathbb{P}_d^k(T)$ satisfies, for all $\mathrm{w} \in \mathbb{P}_d^k(T)$,

$$\boxed{(\mathrm{D}_T^k \underline{\mathbf{t}}_T, \mathrm{w})_T = -(\mathbf{t}_T, \boldsymbol{\nabla} \mathrm{w})_T + \sum_{F \in \mathcal{F}_T} (\mathrm{t}_F \varepsilon_{T,F}, \mathrm{w})_F}$$

where $\varepsilon_{T,F} := \boldsymbol{n}_F \cdot \boldsymbol{n}_{T,F}$

Computation

Requires to invert a SPD matrix of size $N_{k,d} = \binom{k+d}{k}$

Commuting property

The following holds for all $\boldsymbol{t} \in \boldsymbol{S}^+(T)$:

$$\mathrm{D}_T^k \underline{\mathbf{I}}_T^k \boldsymbol{t} = \Pi_T^k(\operatorname{div} \boldsymbol{t})$$

$\rightsquigarrow$ this ensures inf-sup stability for the discretization

Flux reconstruction operator

Local flux reconstruction operator: $\mathbf{F}_T^{k+1} : \underline{\mathbf{S}}_T^k \to \mathbb{M}_T \boldsymbol{\nabla} \mathbb{P}_d^{k+1}(T)$

For all $\underline{\mathbf{t}}_T = (\mathbf{t}_T, \mathrm{t}_{\mathcal{F}_T}) \in \underline{\mathbf{S}}_T^k$, $\mathbf{F}_T^{k+1} \underline{\mathbf{t}}_T := \mathbb{M}_T \boldsymbol{\nabla} \mathrm{z}$, where $\mathrm{z} \in \mathbb{P}_d^{k+1}(T)$ satisfies, for all $\mathrm{w} \in \mathbb{P}_d^{k+1}(T)$,

$$\boxed{(\mathbb{M}_T \boldsymbol{\nabla} \mathrm{z}, \boldsymbol{\nabla} \mathrm{w})_T = -(\mathrm{D}_T^k \underline{\mathbf{t}}_T, \mathrm{w})_T + \sum_{F \in \mathcal{F}_T} (\mathrm{t}_F \varepsilon_{T,F}, \mathrm{w})_F}$$

$\rightsquigarrow$ diffusivity included in reconstruction operator

Computation

Requires to invert a SPD matrix of size $N_{(k+1),d}$

Approximation

For all $v \in H^{k+2}(T)$, letting $\boldsymbol{t} := \mathbb{M}_T \boldsymbol{\nabla} v$, the following holds for all $F \in \mathcal{F}_T$:

$$\|\mathbb{M}_T^{-1/2}(\boldsymbol{t} - \mathbf{F}_T^{k+1} \underline{\mathbf{I}}_T^k \boldsymbol{t})\|_T + h_F^{1/2} \mu_{T,F}^{-1/2} \|(\boldsymbol{t} - \mathbf{F}_T^{k+1} \underline{\mathbf{I}}_T^k \boldsymbol{t}) \cdot \boldsymbol{n}_F\|_F \lesssim \rho_T^{1/2} \mu_{\sharp,T}^{1/2} h_T^{k+1} \|v\|_{k+2,T}$$

Stabilization

$$\boxed{H_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T) := (\mathbb{M}_T^{-1}\mathbf{F}_T^{k+1}\underline{\mathbf{s}}_T, \mathbf{F}_T^{k+1}\underline{\mathbf{t}}_T)_T + J_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T)}$$

Local stabilization bilinear form: $J_T : \underline{\mathbf{S}}_T^k \times \underline{\mathbf{S}}_T^k \to \mathbb{R}$

For all $\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T \in \underline{\mathbf{S}}_T^k$,

$$J_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T) := \sum_{F \in \mathcal{F}_T} \frac{h_F}{\mu_{T,F}} ((\mathbf{F}_T^{k+1}\underline{\mathbf{s}}_T)\cdot \boldsymbol{n}_F - \mathrm{s}_F, (\mathbf{F}_T^{k+1}\underline{\mathbf{t}}_T)\cdot \boldsymbol{n}_F - \mathrm{t}_F)_F$$

Approximation

For all $v \in H^{k+2}(T)$, the following bound holds with $\boldsymbol{t} := \mathbb{M}_T \boldsymbol{\nabla} v$:

$$J_T(\underline{\mathbf{I}}_T^k \boldsymbol{t}, \underline{\mathbf{I}}_T^k \boldsymbol{t})^{1/2} \lesssim \rho_T^{1/2} \mu_{\sharp,T}^{1/2} h_T^{k+1} \|v\|_{k+2,T}$$

Discrete problem

Mixed weak formulation of (1)

Let $\boldsymbol{S} := \boldsymbol{H}(\mathrm{div}, \Omega)$, $V := L^2(\Omega)$. Find $(\boldsymbol{s}, u) \in \boldsymbol{S} \times V$ s.t.

$$\begin{cases} (\mathbb{M}^{-1}\boldsymbol{s}, \boldsymbol{t}) + (u, \mathrm{div}\,\boldsymbol{t}) = 0 & \forall \boldsymbol{t} \in \boldsymbol{S} \\ -(\mathrm{div}\boldsymbol{s}, v) = (f, v) & \forall v \in V \end{cases}$$

Global $\{$hybrid set of flux unknowns$\}$ + $\{$set of potential unknowns $:= \mathbb{P}_d^k(\mathcal{T}_h)\}$

$$\boxed{\underline{\mathbf{S}}_h^k := \mathbb{M}\boldsymbol{\nabla}\mathbb{P}_d^k(\mathcal{T}_h) \times \mathbb{P}_{d-1}^k(\mathcal{F}_h)}$$

Discrete problem: Find $(\underline{\mathbf{s}}_h, \mathrm{u}_{\mathcal{T}_h}) \in \underline{\mathbf{S}}_h^k \times \mathbb{P}_d^k(\mathcal{T}_h)$ s.t.

$$\begin{cases} H_h(\underline{\mathbf{s}}_h, \underline{\mathbf{t}}_h) + (\mathrm{u}_{\mathcal{T}_h}, \mathrm{D}_{\mathcal{T}_h}^k \underline{\mathbf{t}}_h) = 0 & \forall \underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k \\ -(\mathrm{D}_{\mathcal{T}_h}^k \underline{\mathbf{s}}_h, \mathrm{v}_{\mathcal{T}_h}) = (f, \mathrm{v}_{\mathcal{T}_h}) & \forall \mathrm{v}_{\mathcal{T}_h} \in \mathbb{P}_d^k(\mathcal{T}_h) \end{cases}$$

with $H_h(\underline{\mathbf{s}}_h, \underline{\mathbf{t}}_h) := \sum_{T \in \mathcal{T}_h} H_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T)$

Stability

$$\mu_{\sharp,T}^{-1}\|\mathbf{t}_T\|_T^2 + \mu_{\sharp,T}^{-1} \sum_{F \in \mathcal{F}_T} h_F \|\mathrm{t}_F\|_F^2 \lesssim H_T(\underline{\mathbf{t}}_T, \underline{\mathbf{t}}_T) \quad + \quad \text{inf-sup}$$

Error estimates

Theorem (Error estimate for the flux)

Assume $u \in V \cap H^{k+2}(\mathcal{T}_h)$ *and* $\boldsymbol{s} \in \boldsymbol{S} \cap \boldsymbol{S}^+(\mathcal{T}_h)$. *Then,*

$$\|\mathbb{M}^{-1/2}(\boldsymbol{s} - \mathbf{F}_{\mathcal{T}_h}^{k+1}\underline{\mathbf{s}}_h)\| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T}\rho_T h_T^{2(k+1)} \|u\|_{k+2,T}^2 \right\}^{1/2}$$

Theorem (Supercloseness of the potential)

Assume elliptic regularity under the form $\|z(g)\|_{2,\mathcal{T}_h} \lesssim \mu_\flat^{-1}\|g\|$. *Assume* $f \in H^{k+\delta}(\Omega)$, *with* $\delta = 0$ *for* $k \geqslant 1$ *and* $\delta = 1$ *for* $k = 0$. *Then,*

$$\mu_\flat \|\Pi_{\mathcal{T}_h}^k u - \mathrm{u}_{\mathcal{T}_h}\| \lesssim \mu_\sharp^{1/2}\rho^{1/2} h \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T}\rho_T h_T^{2(k+1)} \|u\|_{k+2,T}^2 \right\}^{1/2} + h^{k+2}\|f\|_{k+\delta}$$

Characterization of the solution

Unpatched global hybrid set of flux unknowns

$$\underline{\check{\mathbf{S}}}_h^k := \bigtimes_{T\in\mathcal{T}_h} \underline{\mathbf{S}}_T^k, \qquad \underline{\check{\mathbf{Z}}}_h^k := \left\{ \underline{\check{\mathbf{t}}}_h \in \underline{\check{\mathbf{S}}}_h^k \mid \sum_{T\in\mathcal{T}_F} \check{\mathrm{t}}_{T,F} = 0, \forall F \in \mathcal{F}_h^{\mathrm{i}} \right\}$$

$\rightsquigarrow$ natural isomorphism $\underline{\mathbf{L}}_h^k$ from $\underline{\check{\mathbf{Z}}}_h^k$ to $\underline{\mathbf{S}}_h^k$

Potential-to-flux mapping operator: $\underline{\check{\boldsymbol{\varsigma}}}_h^k : \underline{\mathrm{U}}_h^k \to \underline{\check{\mathbf{S}}}_h^k$

For $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$, $\underline{\check{\boldsymbol{\varsigma}}}_T^k \underline{\mathrm{v}}_T$ satisfies, for all $\underline{\check{\mathbf{t}}}_T \in \underline{\mathbf{S}}_T^k$,

$$H_T(\underline{\check{\boldsymbol{\varsigma}}}_T^k \underline{\mathrm{v}}_T, \underline{\check{\mathbf{t}}}_T) = -(\mathrm{v}_T, \mathrm{D}_T^k \underline{\check{\mathbf{t}}}_T)_T + \sum_{F\in\mathcal{F}_T} (\check{\mathrm{t}}_{T,F}, \mathrm{v}_F)_F$$

$\rightsquigarrow$ There holds: for all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$, $(\mathbf{F}_T^{k+1} \circ \underline{\check{\boldsymbol{\varsigma}}}_T^k)\, \underline{\mathrm{v}}_T = \mathbb{M}_T \boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{\mathrm{v}}_T$

Characterization of the solution

Let $\underline{\tilde{\mathrm{u}}}_h \in \underline{\mathrm{U}}_{h,0}^k$ solve $A_h(\underline{\tilde{\mathrm{u}}}_h, \underline{\mathrm{v}}_h) = (f, \mathrm{v}_{\mathcal{T}_h})$ for all $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k$, with

$$A_h(\underline{\tilde{\mathrm{u}}}_h, \underline{\mathrm{v}}_h) := \sum_{T\in\mathcal{T}_h} (\mathbb{M}_T \boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{\tilde{\mathrm{u}}}_T, \boldsymbol{\nabla} \mathrm{p}_T^{k+1} \underline{\mathrm{v}}_T)_T + \sum_{T\in\mathcal{T}_h} J_T(\underline{\check{\boldsymbol{\varsigma}}}_T^k \underline{\tilde{\mathrm{u}}}_T, \underline{\check{\boldsymbol{\varsigma}}}_T^k \underline{\mathrm{v}}_T)$$

Then, there holds $\underline{\check{\boldsymbol{\varsigma}}}_h^k \underline{\tilde{\mathrm{u}}}_h \in \underline{\check{\mathbf{Z}}}_h^k$ and $(\underline{\mathrm{s}}_h, \mathrm{u}_{\mathcal{T}_h}) = (\underline{\mathbf{L}}_h^k(\underline{\check{\boldsymbol{\varsigma}}}_h^k \underline{\tilde{\mathrm{u}}}_h), \tilde{\mathrm{u}}_{\mathcal{T}_h})$.

Solution strategy

Offline step $\rightsquigarrow$ 4 fully parallelizable and f-independent substeps

- 1 - Compute the divergence reconstruction operator $\mathrm{D}^k_{\mathcal{T}_h}$
  $\rightsquigarrow$ invert $\mathrm{card}(\mathcal{T}_h)$ SPD matrices of size $N_{k,d}$
- 2 - Compute the flux reconstruction operator $\mathbf{F}^{k+1}_{\mathcal{T}_h}$
  $\rightsquigarrow$ invert $\mathrm{card}(\mathcal{T}_h)$ SPD matrices of size $N_{(k+1),d}$
- 3 - Compute the potential-to-flux mapping operator $\underline{\check{\boldsymbol{\varsigma}}}^k_h$
  $\rightsquigarrow$ invert $\mathrm{card}(\mathcal{T}_h)$ SPD matrices of size $N_{k,d} + \mathrm{card}(\mathcal{F}_T) N_{k,(d-1)}$
- 4 - For all $T \in \mathcal{T}_h$, compute the lifting operators $\mathrm{t}^k_T : \mathbb{P}^k_{d-1}(\mathcal{F}_T) \to \mathbb{P}^k_d(T)$ and $\mathrm{d}^k_T : \mathbb{P}^k_d(T) \to \mathbb{P}^k_d(T)$ associated with the bilinear form A_T
  $\rightsquigarrow$ invert $\mathrm{card}(\mathcal{T}_h)$ SPD matrices of size $N_{k,d}$

Online step

- 1 - Given $f \in L^2(\Omega)$, compute its L^2-orthogonal projection $\Pi^k_{\mathcal{T}_h} f$ onto $\mathbb{P}^k_d(\mathcal{T}_h)$
- 2 - Solve the global coercive problem: Find $\tilde{\mathrm{u}}_{\mathcal{F}_h} \in \mathbb{P}^k_{d-1,0}(\mathcal{F}_h)$ s.t.
  $$A_h(\underline{\mathrm{t}}^k_h \tilde{\mathrm{u}}_{\mathcal{F}_h}, \underline{\mathrm{t}}^k_h \mathrm{v}_{\mathcal{F}_h}) = (\Pi^k_{\mathcal{T}_h} f, \mathrm{t}^k_{\mathcal{T}_h} \mathrm{v}_{\mathcal{F}_h}) \qquad \forall \mathrm{v}_{\mathcal{F}_h} \in \mathbb{P}^k_{d-1,0}(\mathcal{F}_h)$$
  $\rightsquigarrow$ solve a linear system of size $\approx \mathrm{card}(\mathcal{F}_h) \times N_{k,(d-1)}$
- 3 - Compute the discrete solution according to $(\underline{\mathbf{s}}_h, \mathrm{u}_{\mathcal{T}_h}) = (\underline{\mathbf{L}}^k_h(\underline{\check{\boldsymbol{\varsigma}}}^k_h \underline{\tilde{\mathrm{u}}}_h), \tilde{\mathrm{u}}_{\mathcal{T}_h})$, with $\underline{\tilde{\mathrm{u}}}_h = (\mathrm{t}^k_{\mathcal{T}_h} \tilde{\mathrm{u}}_{\mathcal{F}_h} + \mathrm{d}^k_{\mathcal{T}_h} \Pi^k_{\mathcal{T}_h} f, \tilde{\mathrm{u}}_{\mathcal{F}_h})$

Outline

Assets of HHO methods

- Capable of handling general polytopal meshes
- Dimension-independent construction
- Arbitrary approximation order (starting from $k = 0$)
- Physical fidelity
  - Local conservation
  - Robustness w.r.t. physical parameters in various situations: heterogeneous/anisotropic diffusion, quasi-incompressible linear elasticity, advection-dominated transport, Stokes flow driven by large irrotational forces, Biot's model of poroelasticity (coupled with DG)...
- Reduced computational cost after static condensation

$$N_{\text{DoFs}}^{\text{HHO}} \approx \frac{1}{2} k^2 \mathrm{card}(\mathcal{F}_h) \qquad \text{vs.} \qquad N_{\text{DoFs}}^{\text{DG}} \approx \frac{1}{6} k^3 \mathrm{card}(\mathcal{T}_h)$$

- Natural offline/online solution strategy: adapted to the multi-query context

THANK YOU FOR YOUR ATTENTION

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